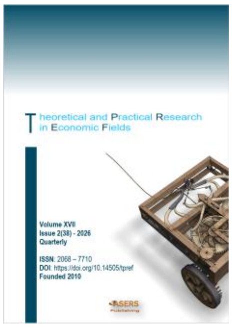
Mathematical Review of the Sukuk-Linked Awqaf Model: A Critical Analysis of Notation Consistency and Formula Validity



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Abstract: This manuscript provides a critical mathematical review of a Sukuk-Linked Awqaf model with the aim of improving notational consistency, analytical clarity, and economic interpretability. The study focuses on correcting weaknesses arising from inconsistent time indexing, redundant equations, misuse of mathematical symbols, and unclear assumptions regarding asset growth, rental income, construction cost, and revenue sharing. By imposing a consistent growth process for waqaf land value, several artificial discount factors are eliminated and present value expressions are derived directly using standard time value of money principles. Key equalities in the original model are reinterpreted as minimum contractual conditions rather than strict identities to better reflect economic uncertainty. The analysis also introduces explicit decision criteria for all stakeholders - Nadzhir, investors, and state-owned enterprises (SOE) - based on present value comparisons. The revised formulation offers a cleaner and more rigorous mathematical foundation for Sukuk-Linked Awqaf and strengthens its relevance as an instrument of Islamic social finance.

Keywords: Sukuk-linked awqaf; Islamic social finance; mathematical modeling; present value; revenue sharing; contract design.

JEL Classification: G15; G21; Z12; C01.

Introduction

The integration of Islamic social finance instruments with capital market mechanisms has attracted increasing scholarly attention in recent years (Kuanova *et al.* 2021, Dirie *et al.* 2024). One prominent example is the attempt to link *awqaf* (Islamic endowment) assets with Sukuk instruments in order to finance public-benefit infrastructure (Qurrata *et al.* 2025). In this context, the Sukuk-Linked Awqaf (SLA) model proposed by Ismal (2024) (Ismal 2024) represents an innovative effort to combine wakaf land, ijarah Sukuk, state-owned enterprises, investors, and tenants within a single financial structure. Conceptually, the model aims to transform unproductive wakaf land into productive social assets through structured financing while maintaining compliance with Islamic jurisprudence (Ismal 2024, Kamaruzaman and Ishak 2023).

Beyond its conceptual and institutional design, the SLA model claims to be supported by a set of mathematical formulations (Ismal 2024). These formulas are intended to represent rental flows, construction costs, Sukuk issuance values, and the investment decisions of both *Nadzhir* (wakaf managers) and Sukuk investors. The author explicitly refers to classical investment theory, especially the net present value (NPV) framework, as developed in standard financial mathematics literature (Benninga 2000, Cvitanic and Zapatero 2024). In principle, such a quantitative foundation is essential: without mathematically consistent and logically valid equations, a financial model cannot be reliably implemented, tested, or compared with alternative structures.

However, in applied mathematics and financial modeling, correctness is not merely a matter of having formulas, but of ensuring that symbols are consistently defined, indices are properly used, summations are meaningful, and discounting reflects the actual timing of cash flows (Czerwinski 2024). Inconsistent notation, misuse of summation indices, or incorrect adaptation of standard formulas - such as future value and present value equations - can lead to results that are mathematically invalid and economically misleading. This is particularly critical when a model is proposed as a policy-relevant or implementable financial structure (Basher and Raboy 2018).

A preliminary reading of the SLA model reveals several potential mathematical issues. These include inconsistent use of time indices (such as n, s, and t), summation expressions that do not depend on their indices, unjustified use of exponential or power forms for linear cash flows, and modifications of NPV formulas that deviate from standard financial mathematics without sufficient theoretical justification. Such issues raise an important question: to what extent can the SLA model be considered a mathematically sound quantitative model, rather than a conceptual framework supported by symbolic but flawed equations?

Therefore, this article aims to conduct a mathematical review of the Sukuk-Linked Awqaf model. The focus is not on its legal, institutional, or Sharia aspects, but strictly on the structure of its mathematical formulations. Specifically, this study critically analyzes:

1. the consistency of symbols and notation,
2. the correctness of summation and indexing,
3. the logical structure of rental, cost, and revenue formulas, and
4. the validity of the modified NPV-based decision criteria.

By applying principles from applied mathematics and financial modeling, this review seeks to clarify which parts of the SLA formulation are mathematically valid, which are inconsistent or incorrect, and how such weaknesses affect the credibility of the model as a quantitative tool. Ultimately, this paper argues that mathematical rigor is not a technical luxury, but a fundamental requirement if innovative Islamic finance models like SLA are to move from conceptual proposals to reliable and implementable financial instruments.

2. Research Methodology

This study employs a critical mathematical review approach. The objective is not to construct a new financial instrument, but to evaluate and refine the mathematical structure of an existing Sukuk-Linked Awqaf model. The review focuses on the consistency of notation, the correctness of formulas, the logical coherence between assumptions and equations, and the economic interpretability of the resulting decision rules. The approach combines narrative explanation of the model, analytical inspection of equations, and critical evaluation of the assumptions that support those equations.

The main object of analysis is a published article that proposes a mathematical formulation for Sukuk-Linked Awqaf and adapts classical accumulation formulas from mathematical finance, including those inspired by Cvitanic and Zapatero (2004) and Ismal (2024). The components under review include investment accumulation formulas, rental income of waqaf land, construction and issuance cost structures, revenue-sharing mechanisms, accumulation of returns for Nadzhir, investors, and SOE, and the decision criteria for each stakeholder.

The data used in this research are secondary data in the form of mathematical expressions, variable definitions, parameter assumptions, and derivation steps presented in the reviewed article. These are supported by references from mathematical finance, time value of money theory, and Islamic finance literature, which serve as benchmarks to assess whether the reviewed formulas align with standard theoretical principles.

Each equation and assumption is evaluated based on four main principles. First, notational consistency is examined to ensure that no symbol carries multiple meanings, indices do not conflict with summation variables, and notation reflects economic interpretation. Second, mathematical correctness is assessed by checking summation bounds, index placement, algebraic manipulation, and redundancy between formulas. Third, logical coherence is evaluated by tracing whether each equation follows from previous assumptions and whether the time structure of construction period, Sukuk tenor, and income period is internally consistent. Fourth, economic interpretability is considered by ensuring that each variable has a clear real-world meaning and that decision rules are reasonable in economic terms.

The analysis begins with content mapping, in which all variables, assumptions, and equations are identified and organized according to their roles in the model. This is followed by mathematical inspection, where each equation is checked for correctness of notation, summation structure, and logical derivation. Key formulas are then compared with standard accumulation models in mathematical finance and with existing Sukuk and waqaf-related models to identify deviations, simplifications, or inconsistencies. When problems are found, the equations are reformulated by presenting the original version, explaining the mathematical issue, and proposing a corrected or simplified version together with its implications.

The review process is conducted sequentially by identifying the structure of the model, tracing the derivation of each equation, detecting conflicts in notation and logic, reformulating problematic equations, and reconstructing the decision criteria for Nadzhir, investors, and SOE. The final outcome of this methodology is a set of formulas that are mathematically consistent, notationally clear, and economically interpretable, which can serve as a reliable foundation for future applied mathematical modeling of Sukuk-Linked Awqaf.

3. Review of the Basic Investment Formula from Cvitanic and Zapatero (2004)

The first formula reviewed in the Sukuk-Linked Awqaf model is taken from Cvitanic and Zapatero (2004) and is intended to represent the accumulation of future investment. In the reviewed article, it is written as:

$$V_t = V_0 + V_1(1+r) + \cdots + V_m(1+r)^m$$

or

$$V_t = V_0 + \sum_{t=1}^{m} V_t\,(1+r)^t$$

This formulation contains a fundamental notational conflict. The symbol $t$is used simultaneously as:

1.the time index of the accumulated value on the left-hand side, and

2.the index of summation on the right-hand side.

This creates a collision of meaning, because t cannot represent both a fixed time point and a running index of summation at the same time. A consistent notation should distinguish between the evaluation time and the summation index. A more appropriate form is:

$$V_T = V_0 + \sum_{i=1}^{m} V_i\,(1+r)^i \qquad (1)$$

where T denotes the final evaluation time and i is the summation index. This avoids ambiguity and preserves logical clarity.

3.1. Deterministic Nature of the Interest Rate

In this formulation, the rate r is treated as a constant (deterministic) parameter. While this is mathematically acceptable for a basic model, it reduces realism in practical applications. In real financial markets, returns are uncertain and time-varying (David and Veronesi 2022, Lim and Brooks 2011, Ren *et al.* 2023, Ang and Timmermann 2012). Therefore, the model implicitly assumes:

- A fixed and known rate of return,
- No risk or volatility,
- Perfect predictability of growth.

This limits the empirical accuracy of the model (Zhou 2018, French 2019). At minimum, this assumption should be explicitly stated, for example: “Assume that the rate of return r is constant and known over the entire investment horizon.”

3.2. Missing Timing Assumptions

The formula also implicitly assumes a specific timing structure of cash flows, but this is not stated. For the formula to be meaningful, one must assume:

- Funds are invested at the beginning of each period,
- They grow during the period at rate r,
- They are withdrawn at the end of the period,
- Then reinvested in the following period.

Without this assumption, the multiplication by $(1+r)^i$ has no clear economic interpretation (Judd 1996, Briggs *et al.* 1997). Therefore, a necessary assumption is: “Each V_i is invested at the beginning of period i, grows at rate $r$during that period, and is evaluated at the end of period i.”

3.3. Review of Equation (2): Present Value Form

The second equation is written as:

$$V_t = V_0 + \frac{V_1}{1+r} + \cdots + \frac{V_m}{(1+r)^m} \qquad (2)$$

or

$$V_t = V_0 + \sum_{t=1}^{m} \frac{V_t}{(1+r)^t}$$

Here, the main error is again notational. The summation index is incorrectly written as t, which clashes with the $t$used on the left-hand side. It should be:

$$V_T = V_0 + \sum_{i=1}^{m} \frac{V_i}{(1+r)^i}.$$

This correction does not change the economic meaning, but it is essential for mathematical consistency.

3.4. Ambiguity of the Symbol V_t

A deeper problem is that the same symbol $V_t$is used in two different meanings:

1. In Equation (1), V_t represents the accumulated future value,
2. In Equation (2), V_t represents a present value of future cash flows.

These are conceptually different objects and should not share the same notation. To avoid confusion:

- Use FV_T for future value at time T,
- Use PV for present value,
- Use V_i only for individual cash flows.

For example:

$$FV_T = \sum_{i=0}^{m} V_i \,(1+r)^i, PV = \sum_{i=0}^{m} \frac{V_i}{(1+r)^i}. \quad (3)$$

If the goal is to highlight simple investment growth, then it is better to write:

$$V_t = V_0(1+r)^t,$$

which clearly represents the growth of a single initial investment.
For cumulative investments, however, a different symbol should be used, such as:

$$A_T = \sum_{i=0}^{m} V_i\,(1+r)^{T-i},$$

so that the distinction between "single-investment growth" and "accumulated multi-period investment" is mathematically and conceptually clear.

4. Review of Equations (4) and (5): Rental Income of Awqaf Land

In the reviewed article, rental income of the awqaf land is defined through Equations (4) and (5) in (Ismal 2024):

$$R_n = (\theta A_0)^1 + (\theta A_0)^2 + \cdots + (\theta A_0)^n \quad (4)$$

$$R_n = \sum_{1}^{n} (\theta\, A_0)^n \quad (5)$$

These two equations are mathematically redundant and, more importantly, both are incorrectly formulated.

4.1. Constant Market Value Assumption

The article defines A_0 as the current market value of the awqaf asset (Ismal 2024). Implicitly, this suggests that $A_0$is treated as constant over time. If A_0 is indeed assumed constant and the rental rate θ is fixed, then the rental payment each period is simply:

$$\text{Rental per period} = \theta A_0.$$

Therefore, the accumulated rental over $n$periods should be:

$$R_n = \theta A_0 + \theta A_0 + \cdots + \theta A_0 (n \text{ terms}),$$

which simplifies to:

$$R_n = n\theta A_0.$$

In this case, there is no need to use summation notation, because the expression does not depend on any index. Thus, Equations (4) and (5) should be replaced by a single, simple formula:

$$R_n = n\theta A_0.$$

4.2. Unrealistic Constancy of Market Value

However, in real applications, the market value of land rarely remains constant (Regan *et al.* 2015 Pagourtzi *et al.* 2003, Xie *et al.* 2002). While the rental rate θ may be contractually fixed, the underlying asset value usually changes over time. Therefore, a more realistic assumption is:

- θ is constant,
- The land value evolves over time:

$$A_i = A_0(1+r)^i,$$

where r is the growth rate of land value.
Under this assumption, rental in period i becomes:

$$\text{Rental in period } i = \theta A_i = \theta A_0(1+r)^i.$$

Then, total rental income over $n$periods is:

$$R_n = \sum_{i=1}^{n} \theta\, A_0(1+r)^i.$$

This is a geometric series and can be simplified as:

$$R_n = \theta A_0 \sum_{i=1}^{n}(1+r)^i = \theta A_0 \cdot \frac{(1+r)^{n+1} - (1+r)}{r}, r \neq 0.$$

5. Review of Construction Cost and Issuance Cost Formulas

5.1. Clarification of Time Structure: Relation between n, s, and t

In the reviewed article (Ismal 2024), three-time parameters are used:

- n = leasing period of awqaf land by the SOE,
- t = construction period,
- s = Sukuk tenor.

However, the article only states that $s < n$, without clearly explaining the economic meaning of this relation. In fact, the structure of the model implies:

- During the construction period t, the SOE already leases the awqaf land and pays rent,
- After construction, during Sukuk tenor s, the building is rented to tenants and generates income,
- Therefore, the total leasing period should satisfy:

$$n = s + t.$$

This relation is essential for time consistency (Calvo 1978, Rafay *et al.* 2017, Nurhanifah 2024). Without explicitly stating $n = s + t$, the interaction between rental payments, construction costs, and Sukuk tenor becomes mathematically ambiguous.

5.2. Review of Equations (6) and (7): Construction Cost

The reviewed article (Ismal 2024) presents:

$$K_t = (\beta A_s)^1 + (\beta A_s)^2 + \cdots + (\beta A_s)^t \quad (6)$$

or

$$K_t = \sum_1^t (\beta\, A_s)^t \quad (7)$$

Because the word "or" is used, both equations are intended to represent the same quantity. However, both are mathematically incorrect and redundant.

If β is a fixed proportion and A_s is the value of the asset used as the construction reference, then construction cost per period is:

$$\beta A_s.$$

Hence, total construction cost over t periods is:

$$K_t = \beta A_s + \beta A_s + \cdots + \beta A_s (t \text{ terms}) = t\beta A_s.$$

There is no need for summation notation, because the right-hand side does not depend on any index.

5.3. Allowing Asset Value Growth

If the asset value used for construction is assumed to evolve over time (Acciaio *et al.* 2025), then one may assume:

$$A_s = A_0(1+r)^s, \text{with } s = n - t.$$

Then construction cost becomes:

$$K_t = t\beta A_s = t\beta A_0(1+r)^{n-t}.$$

This formulation is mathematically consistent and economically interpretable.

5.4. Review of Other Costs (Ismal 2024)

The article defines "other costs" using the symbol S, while at the same time using s to denote Sukuk tenor. This creates unnecessary confusion, because S and $s$are visually and conceptually too similar.

It is better to denote "other costs" by O (Critical that O is not yet used in the model). Then:

$$O_s = O + O + \cdots + O(s \text{ terms}) = sO.$$

Thus, Equations (8) and (9) in (Ismal 2024) can be replaced by:

$$O_s = sO.$$

5.5. Cost and Issuance Value in (Ismal 2024)

The article defines:

$$C_s = \text{total cost}, I_s = \text{Sukuk issuance value},$$

and then states:

$$I_s = C_s.$$

This makes one of the two symbols redundant. It is sufficient to use only one main symbol. A clearer choice is:

I_s: value of Sukuk issuance, where I stands for "Issuance" and s for Sukuk tenor.

Then the issuance value can be written directly as:

$$I_s = \sum_{i=1}^{n} \theta A_0 + \sum_{i=1}^{t} \beta A_s + \sum_{i=1}^{s} O.$$

Since each term is constant within its period, this simplifies to:

$$I_s = n\theta A_0 + t\beta A_s + sO.$$

If asset value grows as $A_s = A_0(1+r)^s$, then:

$$I_s = n\theta A_0 + t\beta A_0(1+r)^s + sO.$$

6. Review of Revenue Coefficient $\gamma$and Income Timing

In the reviewed model (Ismal 2024), θ and β are clearly defined as proportional parameters:

- θ = rental proportion of awqaf land,
- β = construction cost proportion.

However, the parameter γ is less clearly defined. Conceptually, γ represents the business return coefficient of the building constructed on waqaf land (Ismal 2024, Zafar and Jafar 2025). Therefore, unlike θ and β, which are cost-related proportions, γmeasures operational performance.

From a mathematical and economic perspective, γ should satisfy:

$$0 \leq \gamma < \infty.$$

Interpretation:
- $\gamma = 0$: the business generates no profit,
- $\gamma > 0$: the business is profitable,
- In practice, one may also allow $\gamma < 0$ to represent losses.

However, if $\gamma < 0$, the model must clearly specify who bears the loss. A consistent assumption is:
- Losses are borne by Sukuk investors and SOE (Hamzah *et al.* 2018, Klein *et al.* 2018, Mohd Roslen *et al.* 2025, Ahmed *et al.* 2018),
- Nadzhir does not bear operational losses, in line with the social and protected nature of waqaf assets.

This risk-sharing rule is not clearly stated in the reviewed article and should be made explicit.

6.1. Ambiguity in Construction Financing and Time Consistency

The article is unclear about whether construction costs are initially financed by the SOE or directly from Sukuk proceeds (Ismal 2024). This ambiguity affects time consistency.

If construction is financed by Sukuk from the start, then income generation cannot begin until construction ends, and the relation between time variables becomes unclear. To maintain consistency, it is more realistic to assume:
- Construction cost is initially borne by the SOE,
- Sukuk is issued after the building is ready and operational,
- Sukuk proceeds are used mainly for refinancing and income distribution.

This structure is similar to livestock or agriculture Sukuk, where assets must exist and operate before Sukuk returns are distributed (Ashfahany *et al.* 2023, Alswaidan *et al.* 2017, Hasan *et al.* 2019).

Under this assumption:
- t = construction period,
- s = income-generating period,
- $n = t + s$ = total leasing period of awqaf land by SOE,
- Income is received only after construction ends.

Thus, the correct income period is not "$s - t$" as written in the article, but:

$$\text{Income period} = n - t = s.$$

6.2. Correcting the Income Formula

The article writes income as:

$$\pi_{s-t} = \sum \gamma A_s^{(s-t)},$$

which is both notationally and conceptually incorrect. With the corrected time structure:

- Income is generated over $n - t$periods,
- Return per period is γA_s,

so total income becomes:

$$\pi_{n-t} = \gamma A_s + \gamma A_s + \cdots + \gamma A_s (n - t \text{ terms}) = (n - t)\gamma A_s.$$

There is no need for summation notation if the term is constant.

6.3. Revenue Sharing

Revenue is divided among three parties: Nadzhir, Sukuk investors, and SOE (Rafay, Sadiq, and Ajmal, 2017). To avoid confusion with the summation index i, Sukuk investors' income should be denoted by π_v(from *value holder*), not π_i.

Thus:

$$\pi_{n-t} = \pi_z + \pi_v + \pi_e,$$

with equal sharing:

$$\pi_z = \pi_v = \pi_e = \frac{1}{3}(n - t)\gamma A_s.$$

If losses are allowed ($\gamma < 0$), then:
- $\pi_z = 0$ (Nadzhir protected),

- Loss is shared between SOE and Sukuk investors,

which must be specified as an explicit assumption.

7. Review of Nadzhir's Accumulation and Investment Decision

At the end of the Sukuk tenor and land leasing period, the building constructed on waqaf land is transferred to Nadzhir (Ismal 2024), Alam *et al.* 2024, Adniyah 2025). During the Sukuk period, Nadzhir gradually "pays" for the building using two sources:

1. Rental payment from the waqaf land paid by SOE,
2. Nadzhir's share of business income generated from the building.

Thus, Nadzhir's effective payment for the building is:

$$R_n + \pi_z.$$

7.1. Relation between Construction Cost and Nadzhir's Payment

In the reviewed article (Ismal 2024), it is imposed that:

$$K_t = R_n + \pi_z.$$

This equality is not necessarily true in general. In reality:

If the business return coefficient γ is very large, then $\pi_z$may be large and:

$$K_t < R_n + \pi_z.$$

If the business is weak, π_z may be small, and:

$$K_t > R_n + \pi_z.$$

However, for pricing and contract design, it is reasonable to impose a minimum condition (Hameed and Bashir 2002):

$$K_t \le R_n + \pi_z,$$

and more precisely, the rental rate $\theta$and revenue-sharing scheme should be set such that:

$$K_t = R_n + \pi_z \text{ (as a minimum target).}$$

This ensures that Nadzhir can fully acquire the building by the end of the Sukuk period.

7.2. Final Asset Value of Nadzhir

The article (Ismal 2024) defines:

$$P_z = A_n + K_t.$$

However, since Nadzhir actually pays for the building through $R_n + \pi_z$, the more accurate formulation is:

$$P_z = A_n + (R_n + \pi_z).$$

This reflects the fact that Nadzhir's final asset consists of:

- The waqaf land valued at A_n, and
- The building, effectively paid through accumulated rental and income shares.

7.3. Present Value of Nadzhir's Receivables

To evaluate Nadzhir's decision, the present value of its final wealth must be computed consistently (Sulaiman and Alhaji Zakari 2019).

Assume land value evolves as:

$$A_i = A_0(1+r)^i.$$

Then:

$$A_n = A_0(1+r)^n.$$

Thus:

$$\text{PV of land value} = \frac{A_n}{(1+r)^n}.$$

Rental over $n$periods (with constant A_0) is:

$$R_n = n\theta A_0, \quad \text{PV of rental} = \frac{n\theta A_0}{(1+r)^n}.$$

Business income share for Nadzhir over income period $n - t$ (with $s = n - t$) is:

$$\pi_z = \frac{1}{3}(n-t)\gamma A_s, \qquad A_s = A_0(1+r)^s.$$

Hence:

$$\text{PV of income share} = \frac{1}{3}\sum_{i=1}^{n-t}\frac{\gamma A_s}{(1+r)^s}.$$

Since the term is constant, this simplifies to:

$$PV_{\pi_z} = \frac{1}{3}\frac{(n-t)\gamma A_s}{(1+r)^s}.$$

Therefore, Nadzhir's total present value is:

$$PV_z = \frac{A_n}{(1+r)^n} + \frac{n\theta A_0}{(1+r)^n} + \frac{1}{3}\frac{(n-t)\gamma A_s}{(1+r)^s}, s = n - t.$$

7.4. Correction of Assumptions

The reviewed article introduces abstract discount factors such as:

$$PV_{A_n}, PV_{R_n}, PV_{\pi_z},$$

without consistently linking them to the assumed growth:

$$A_i = A_0(1+r)^i.$$

With this assumption, there is no need to introduce separate symbolic discount factors. The correct expressions are directly:

$$PVA_n = \frac{A_n}{(1+r)^n}, PVR_n = \frac{n\theta A_0}{(1+r)^n}, PV\pi_z = \frac{1}{3}\sum_{i=1}^{n-t}\frac{\gamma A_s}{(1+r)^s}, \; s = n - t.$$

Thus, Nadzhir's decision criterion becomes:

$$PV_z = PVA_n + PVR_n + PV\pi_z.$$

Nadzhir is willing to lease the waqaf land if:

$$PV_z > A_0,$$

meaning that the present value of future benefits exceeds the current market value of the waqaf asset.

7.5. Correction of Inequality Notation (with Consistent Subscript)

In the reviewed article (Ismal 2024), the decision condition for Nadzhir is written using an angle bracket notation, for example:

$$PV_n\rangle A_0$$

or a similar symbol to represent comparison. This is mathematically incorrect. Angle brackets are not used to denote inequalities; they are commonly used for:

- Inner products in vector spaces,
- Dual pairings in functional analysis,
- Bra–ket notation in physics.

Using angle brackets to represent "greater than" or "less than" is a notational error and can seriously confuse readers, especially those with a mathematical background.

Moreover, since the decision maker is Nadzhir, and the subscript $n$has already been reserved for time horizon, the present value for Nadzhir should be written as PV_z, not PV_n.

7.6. Correct Form of the Inequality

The correct decision rule for Nadzhir should be written using standard inequality symbols:

$$PV_z > A_0.$$

This means that Nadzhir will agree to lease the waqaf land if the present value of all future benefits exceeds the current market value of the waqaf asset.

7.7. Can It Be Written as $A_n > \cdots$?

Yes, it is possible to rewrite the condition in terms of A_n, provided we use the assumed growth:

$$A_n = A_0(1+r)^n.$$

From:

$$PV_z > A_0,$$

and:

$$PV_z = \frac{A_n}{(1+r)^n} + \frac{n\theta A_0}{(1+r)^n} + \frac{1}{3}\frac{(n-t)\gamma A_s}{(1+r)^s}, s = n-t,$$

multiply both sides by $(1+r)^n > 0$, so the inequality direction is preserved:

$$A_n + n\theta A_0 + \frac{1}{3}(n-t)\gamma A_s(1+r)^{n-s} > A_0(1+r)^n.$$

Since $A_0(1+r)^n = A_n$, this simplifies to:

$$n\theta A_0 + \frac{1}{3}(n-t)\gamma A_s(1+r)^{n-s} > 0.$$

Thus, the condition can be written without present value notation as:

$$n\theta A_0 + \frac{1}{3}(n-t)\gamma A_s(1+r)^{n-s} > 0.$$

This form is always satisfied if:

$$\theta \geq 0, \gamma \geq 0.$$

Therefore, economically meaningful decision making still relies more clearly on the PV formulation:

$$PV_z > A_0,$$

rather than forcing it into a less intuitive inequality involving A_n.

7.8. Clean Final Version for the Paper

The inequality should be written as:

$$PV_z > A_0.$$

If one insists on eliminating present value notation, it can be equivalently written as:

$$A_n + n\theta A_0 + \frac{1}{3}(n-t)\gamma A_s(1+r)^{n-s} > A_0(1+r)^n,$$

which simplifies to:

$$n\theta A_0 + \frac{1}{3}(n-t)\gamma A_s(1+r)^{n-s} > 0.$$

However, for clarity and economic interpretation, the simplest and cleanest decision rule remains:

$$PV_z > A_0.$$

This avoids notational abuse, respects the distinction between time index (n) and agent index (z), and aligns with standard mathematical and economic conventions.

8. Correction for Investor's Accumulation and Notation

In the reviewed article, the payoff for investors is sometimes written using subscript i. This is problematic because:

- The symbol $i$is already used as a running index in summation notation $\sum_{i=1}^{n}$,
- Using the same symbol both as an index and as an agent identifier creates notational ambiguity.

Therefore, the payoff for investors should be written with subscript v(from *investor* or *value holder*), not i. Thus, the correct notation is:

$$P_v = I_s + \pi_v,$$

not $P_i = I_s + \pi_i$.

8.1. Investor's Accumulation Formula

From previous corrections, the Sukuk issuance value is:

$$I_s = n\theta A_0 + t\beta A_s + sO,$$

and investor's income share is:

$$\pi_v = \frac{1}{3}(n-t)\gamma A_s, s = n - t.$$

Therefore, investor's total accumulation at maturity is:

$$P_v = n\theta A_0 + t\beta A_s + sO + \frac{1}{3}(n-t)\gamma A_s, s = n - t.$$

There is no need to use summation notation here because each term is constant over its period.

8.2. Inconsistency in the Reviewed Article

The article writes something like:

$$P_i = n\theta A_0 + t\beta A_s + sO + \frac{1}{3}\sum_{i=1}^{n-t}\gamma A_s,$$

which is inconsistent because:

1. The same symbol $i$is used both as agent index and summation index,
2. The summation is unnecessary since the summand does not depend on i,
3. It contradicts the stated assumption that $s = n - t$.

The correct and consistent form is:

$$P_v = n\theta A_0 + t\beta A_s + sO + \frac{1}{3}(n-t)\gamma A_s, s = n - t.$$

Assume:

$$A_i = A_0(1+r)^i.$$

Then:

$$A_s = A_0(1+r)^s, s = n - t.$$

Thus, the present value of investor's payoff is:

$$PV_v = \frac{P_v}{(1+r)^s}.$$

So:

$$PV_v = \frac{n\theta A_0 + t\beta A_s + sO + \frac{1}{3}(n-t)\gamma A_s}{(1+r)^s}, s = n - t.$$

This is the correct present value formulation for investors, with:

- Clear distinction between time index $n$and agent index v,
- No collision between summation index and agent notation,
- Consistent use of $s = n - t$.

9. Correction of Investor Decision Inequality

In the reviewed article (Ismal 2024), the investor decision condition is again written using angle brackets, something like:

$$PV_v \rangle I_s$$

to indicate comparison. This is a notational error. As explained earlier, angle brackets are not used for inequalities; they are reserved for inner products or pairings.
The correct form must use standard inequality symbols.

9.1. Correct Investor Decision Rule

Investors will be willing to buy the Sukuk if the present value of what they receive exceeds what they pay, i.e.:

$$PV_v > I_s.$$

Not:

$$PV_v \rangle I_s.$$

9.2. Substituting the Expressions

From previous corrections:

$$I_s = n\theta A_0 + t\beta A_s + sO,$$
$$P_v = n\theta A_0 + t\beta A_s + sO + \frac{1}{3}(n-t)\gamma A_s, s = n - t,$$
$$PV_v = \frac{P_v}{(1+r)^s}.$$

Thus, the correct inequality becomes:

$$\frac{n\theta A_0 + t\beta A_s + sO + \frac{1}{3}(n-t)\gamma A_s}{(1+r)^s} > n\theta A_0 + t\beta A_s + sO.$$

This is the mathematically correct form of the investor decision condition, using proper inequality notation instead of angle brackets.
Multiply both sides by $(1+r)^s > 0$(the direction of inequality is preserved):

$$n\theta A_0 + t\beta A_s + sO + \frac{1}{3}(n-t)\gamma A_s > (n\theta A_0 + t\beta A_s + sO)(1+r)^s.$$

Move the common terms to one side:

$$\frac{1}{3}(n-t)\gamma A_s > (n\theta A_0 + t\beta A_s + sO)((1+r)^s - 1).$$

Hence, the simplified and clean form of the investor's inequality is:

$$\boxed{\frac{1}{3}(n-t)\gamma A_s > (n\theta A_0 + t\beta A_s + sO)((1+r)^s - 1), s = n - t.}$$

Interpretation:
- The left-hand side represents the additional return obtained by investors from profit sharing,
- The right-hand side represents the opportunity cost of tying funds in Sukuk for $s$periods,
- Investors are willing to participate if the extra return exceeds the time value of money they sacrifice.

10. Decision Criterion for SOE

In the reviewed model (Ismal 2024), decision rules are formulated for Nadzhir and for Sukuk investors, but no explicit decision criterion is provided for the SOE. This is a conceptual gap, because SOE is a key economic agent that:
- Bears (or pre-finances) construction cost,
- Pays land rent to Nadzhir during the whole leasing period,
- Shares business profit or loss,
- Returns the building to Nadzhir at maturity.

Therefore, SOE also needs a rational economic decision rule (Xie et al., 2002; Zafar and Jafar, 2025; Zhou, 2018).

10.1. Cash Flows of SOE

From the corrected structure:
Outflows of SOE:
1. Construction cost during construction period:

$$K_t = t\beta A_s.$$

2. Land rent during entire leasing period:

$$R_n = n\theta A_0.$$

3. SOE's share of operational result (if loss, SOE bears part of it):

$$\pi_e = \frac{1}{3}(n-t)\gamma A_s.$$

(Note: if $\gamma < 0$, this becomes a cost.)

Inflows of SOE:

1. SOE's share of business income:

$$\pi_e = \frac{1}{3}(n-t)\gamma A_s (\text{if } \gamma > 0),$$

2. Possibly refinancing from Sukuk issuance if construction is refinanced.

For simplicity, assume construction is pre-financed by SOE and later refinanced by Sukuk issuance I_s. Then SOE receives:

$$I_s = n\theta A_0 + t\beta A_s + sO.$$

But since most of I_s is used to pay construction and other costs, the true economic gain of SOE mainly comes from its profit share π_e.

Net Payoff of SOE

Thus, SOE's net payoff at maturity can be written as:

$$P_e = \pi_e - K_t - R_n + I_s.$$

Substitute:

$$P_e = \frac{1}{3}(n-t)\gamma A_s - t\beta A_s - n\theta A_0 + (n\theta A_0 + t\beta A_s + sO).$$

Simplify:

$$P_e = \frac{1}{3}(n-t)\gamma A_s + sO.$$

If "other costs" O are paid by investors and not by SOE, then SOE's economic benefit reduces to:

$$P_e = \frac{1}{3}(n-t)\gamma A_s.$$

Present Value of SOE's Payoff

Income is received over the operational period $s = n - t$. Thus, present value for SOE is:

$$PV_e = \frac{P_e}{(1+r)^s} = \frac{\frac{1}{3}(n-t)\gamma A_s}{(1+r)^s}.$$

10.2. Decision Rule for SOE

SOE will participate in the scheme if its present value is non-negative:

$$PV_e \geq 0.$$

Equivalently:

$$\frac{1}{3}(n-t)\gamma A_s \geq 0.$$

Thus:

- If $\gamma \geq 0$, SOE has no economic disincentive to participate,
- If $\gamma < 0$, SOE will suffer losses and will rationally reject the project unless compensated by other contractual benefits.

Hence, the SOE decision criterion can be stated as:

$$\boxed{PV_e \geq 0 \quad \Leftrightarrow \quad \gamma \geq 0.}$$

10.3. Interpretation

- Nadzhir's decision: $PV_z > A_0$,
- Investor's decision: $PV_v > I_s$,
- SOE's decision: $PV_e \geq 0$.

All three agents now have clear, mathematically consistent participation criteria.

Conclusions and Further Research

This study has critically reviewed the mathematical structure of the Sukuk-Linked Awqaf model with the main objective of improving notational clarity, mathematical consistency, and economic interpretability. By systematically examining each assumption and equation, the study shows that several weaknesses in the original formulation arise not from the economic idea itself, but from inconsistent notation, redundant formulas, unclear time indexing, and the misuse of mathematical symbols such as angle brackets for inequalities.

The analysis demonstrates that many equations can be significantly simplified once consistent growth assumptions are imposed, especially for the evolution of waqaf land value. When the land value is assumed to follow a deterministic growth process, several artificial discount symbols become unnecessary, and present value expressions can be written directly in terms of standard time value of money principles. This not only reduces redundancy but also improves transparency for readers from both mathematical and economic backgrounds.

Another important result is the reinterpretation of key equalities as minimum pricing conditions rather than strict identities. In particular, the relation between construction cost and Nadzhir's accumulated payments should be treated as a contractual target or lower bound, not as an unconditional equality. This perspective better reflects real economic uncertainty in business income and rental performance and avoids unrealistic assumptions embedded in deterministic equalities.

The study also highlights the importance of clearly distinguishing time indices for different roles, such as summation indices, period counters, and stakeholder identifiers. Misuse of indices leads to confusion and potential misinterpretation of formulas. By correcting these issues, the revised model provides a cleaner structure for rental income, construction cost, revenue sharing, and accumulation of returns for Nadzhir, investors, and SOE.

Furthermore, this study extends the reviewed model by explicitly formulating decision criteria for all stakeholders, including Nadzhir, investors, and SOE. Each decision rule is expressed using standard inequality notation and linked directly to present value analysis. This ensures that every party's participation in the Sukuk-Linked Awqaf scheme is justified through transparent and economically meaningful conditions.

Overall, this manuscript does not reject the original Sukuk-Linked Awqaf concept, but strengthens it by providing a mathematically rigorous and notationally consistent framework. The revised formulation offers a clearer foundation for future theoretical development, empirical calibration, and policy application of Sukuk-Linked Awqaf as an instrument of Islamic social finance. Future research can build on this framework by introducing uncertainty, stochastic returns, and empirical data to test the feasibility and performance of the model in real-world implementations.

Declarations

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