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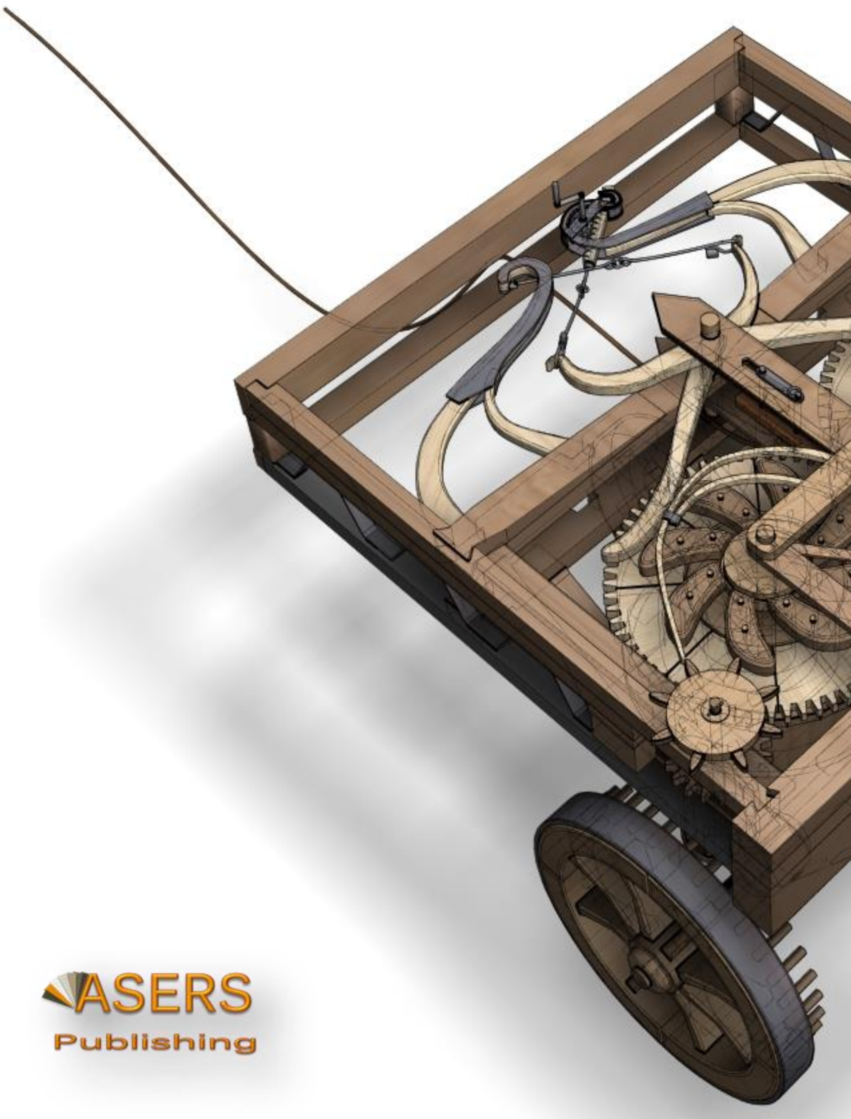

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FUZZINESS AND STATISTICS – MATHEMATICAL MODELS FOR UNCERTAINTY

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Abstract:

Real data from continuous quantities, considered under different models in economic theory, cannot be measured precisely. As a result, measurement results cannot be accurately represented by real numbers, as they contain different kinds of uncertainty. Beside errors and variability, individual measurement results are more or less fuzzy as well. Therefore, real data have to be described mathematically in an adequate way. The best up-to-date models for this are so-called fuzzy numbers, which are special fuzzy subsets of the set of real numbers. Based on this description, statistical analysis methods must be generalized to the situation of fuzzy data. This is possible and will be explained here for descriptive statistics, inferential statistics, objective statistics, and Bayesian inference.

Keywords: Bayesian analysis; descriptive statistics; fuzzy information; fuzzy numbers; statistical inference

JEL Classification: C11; C13; C15

1. Introduction

Measurement results of continuous variables are often clouded with uncertainty. In addition, many data are not exact numbers but more or less fuzzy. This type of uncertainty differs from measurement errors. In fact, the obtained data are generally associated with various types of uncertainty. There are multiple components to uncertainty in economic analyses, and it is a challenge for researchers to characterize the full nature and magnitude of these components (Hansen 2017). This challenge should be approached with caution for an accurate data analysis. Data with uncertainty (fuzzy data) are common in economic analysis, *i.e.* economic indicators such as measures of trade, labor force, and stock market. In these cases, the data to be reviewed and/or further analyzed are often presented with considerable uncertainties. Nevertheless, such data, despite their uncertainties, are essential for decisions and often critical. Different methods have been developed for analyzing or correcting results from the use of incomplete data (Krasker 1983). The description of fuzzy data and their statistical analysis also form an active field of research. The most up-to-date mathematical model to describe the fuzziness is fuzzy numbers and their characterizing functions (Viertl 2015).

In this contribution, the generalized statistical methods to handle fuzzy data, common in economic analysis, are described. In section 2, definition of fuzzy data, fuzzy probability densities, and fuzzy-valued functions are explained. Definition of fuzzy sample and some of generalized methods for measures of location (central tendency) as well as dispersion are described in section 3. Some additional useful descriptive statistics for fuzzy data are explained in section 4. In section 5, inferential statistics based on fuzzy information are described. In section 6, the generalized method for Bayesian Inference with fuzzy data is presented. An open-end and critical research area, *i.e.* fuzzy stochastic processes, is suggested in section 7. The contribution is concluded with final remarks in section 8.

2. Fuzziness

Many important economic information is not obtained as precise data, but rather imprecise, e.g. high income, low interest rate, good quality, and many more. A modern quantitative description of such linguistic variables is fuzzy sets. Hence, the occurring uncertainty can be modelled using the theory of fuzzy random functions (Möller 2009). A fuzzy subset of a given set M is a generalization of the *indicator function* $I_A(\cdot)$ of a classical subset $A \subseteq M$. These generalized functions are called membership functions $\xi(\cdot)$, which are functions $\xi: M \to [0,1]$. The value $\xi(x)$ for $x \in M$ is the degree to which x belongs to the fuzzy set defined by $\xi(\cdot)$.

Remark 2.1 Indicator functions are special forms of membership functions obeying $I_A: M \to \{0,1\}$, *i.e.* they allow only two possible values of 0 and 1.

General membership functions can assume all values from the closed unit interval [0,1], however. Fuzzy subsets of a universal set M are determined by a family of classical subsets of M, *i.e*. the so-called δ-cuts.

Definition 2.1 Let $\xi(\cdot)$ be the membership function of a fuzzy subset of the universal set M. For all $\delta \in (0,1]$, the δ-cut $C_\delta[\xi(\cdot)]$ is defined by $C_\delta[\xi(\cdot)] := \{x \in M: \xi(x) \geq \delta\}$.

Now the following representation lemma is valid (Viertl 2011).

Lemma 2.1 For every membership function $\xi(\cdot)$, the following holds true:

$$\xi(x) = max\{\delta.\, I_{C_\delta[\xi(\cdot)]}(x_0): \delta \in [0,1]\} \quad \forall x \in M$$

Proof: For fixed $x_0 \in M$ and $\delta \in [0,1]$, we have

$$\delta \cdot I_{C_\delta[\xi(\cdot)]}(x_0) = \delta \cdot I_{\{x\,:\,\xi(x) \geq \delta\}}(x_0) = \begin{Bmatrix} \delta \text{ for } \xi(x_0) \geq \delta \\ 0 \text{ for } \xi(x_0) < \delta \end{Bmatrix}.$$

Therefore, we obtain for every $\delta \in [0,1]$, $\delta \cdot I_{C_\delta[\xi(\cdot)]}(x_0) \leq \xi(x_0)$, and further,

$sup\{\delta \cdot I_{C_\delta[\xi(\cdot)]}(x_0): \delta \in [0,1]\} \leq \xi(x_0)$. On the other hand, we have for $\delta_0 = \xi(x_0)$:

$\delta_0 \cdot I_{C_{\delta_0}[\xi(\cdot)]}(x_0) = \delta_0$ and, therefore, $sup\{\delta \cdot I_{C_\delta[\xi(\cdot)]}(x): \delta \in [0,1]\} \geq \delta_0$,

which implies $sup\{\delta \cdot I_{C_\delta[\xi(\cdot)]}(x_0): \delta \in [0,1]\} \leq \xi(x_0) = max\{\delta \cdot I_{C_\delta[\xi(\cdot)]}(x_0): \delta \in [0,1]\} = \delta_0$.

Remark 2.2: Fuzzy sets are determined by the family of their δ-cuts $C_\delta[\xi(\cdot)]; \delta \in (0,1]$. The family of δ-cuts is a nested family of subsets of M, *i.e.* $\delta_1 < \delta_2 \Rightarrow C_{\delta_1} \supseteq C_{\delta_2}$. The question is whether every nested family of subsets of M are the δ-cuts of a fuzzy set in M. There are counterexamples, but the following construction lemma holds:

Lemma 2.2 Let $(A_\delta; \delta \in (0,1])$ be a nested family of subsets of a given set M. Then, a fuzzy subset of M is generated, whose membership function is defined by

$$\xi(x) = sup\{\delta \cdot I_{A_\delta}(x_0): \delta \in [0,1]\} \qquad \forall x \in M,$$

for which the following holds true:

$$C_\delta[\xi(\cdot)] = A_\delta \text{ IFF } \bigcap_{\beta<\delta} A_\beta = A_\delta$$

Proof: First we extend the family $(A_\delta; \delta \in (0,1])$ by the element $A_0 = M$. Then, the proof is in the following three steps:

(1) $A_\delta \subseteq C_\delta[\xi(\cdot)] \quad \forall \delta \in (0,1]$:

For $x \in A_\delta$, we have $\delta \cdot I_{A_\delta}(x) = \delta$ and, thus, $sup\{\beta \cdot I_{A_\beta}(x): \beta \in (0,1]\} \geq \delta$. By definition of $\xi(\cdot)$, we have $\xi(x) \geq \delta$ and, therefore, $x \in C_\delta[\xi(\cdot)]$.

(2) $A_\delta = \bigcap_{\beta<\delta} A_\beta \Rightarrow C_\delta[\xi(\cdot)] = A_\delta$:

For $x \notin A_\delta$, from $A_\delta = \bigcap_{\beta<\delta} A_\beta$, we know there exists $\alpha < \beta$ with $x \notin A_\alpha$, and by the nested structure of the generating family $(A_\delta; \delta \in (0,1])$, we know that $x \notin A_\beta \; \forall \beta \in (\alpha, 1]$. Therefore,

$\xi(x) = sup\{\beta \cdot I_{A_\beta}(x): \beta \in (0,1]\} \leq \alpha < \delta$, and therefore $x \notin C_\delta[\xi(\cdot)]$.

(3) $C_\delta[\xi(\cdot)] = A_\delta \Rightarrow A_\delta = \bigcap_{\beta<\delta} A_\beta$:

$A_\delta \subseteq \bigcap_{\beta<\delta} A_\beta$ holds by the nested structure of $(A_\delta; \delta \in (0,1])$. For $x \notin C_\delta[\xi(\cdot)]$, assuming $A_\delta = C_\delta[\xi(\cdot)]$, we obtain $\xi(x) = sup\{\beta \cdot I_{A_\beta}(x): \beta \in (0,1]\} < \delta$. Choosing $\alpha \in (\xi(x), \delta)$, we have $x \in A_\delta$ and, therefore $x \notin \bigcap_{\beta<\delta} A_\beta$.

Remark 2.3 For a given nested family of subsets of M, the generated fuzzy subset contains δ-cuts which are equal to the generating sets as far as possible. See also (Zadeh 1965 and Dubois 1987) for related mathematical definitions.

2.1. Fuzzy Data

Measurement data from continuous quantities are always more or less imprecise, *i.e.* they cannot be represented by precise numbers. Therefore, a more general concept than real numbers is necessary. The best up-to-date such models are so-called fuzzy numbers.

Definition 2.2 A fuzzy number x^* is a fuzzy subset of the set of real numbers $\mathbb{R}$, whose membership function $\xi(\cdot)$ obeys the following:

(1) supp$[\xi(\cdot)]$ is a bounded set, i.e. supp$[\xi(\cdot)] \subseteq [a,b]$ with $-\infty < a < b < \infty$.

(2) $C_\delta[\xi(\cdot)]$ is a finite union of compact intervals, i.e.

$C_\delta[\xi(\cdot)] = \bigcup_{j=1}^{k_\delta} [a_{\delta,j}, b_{\delta,j}] \neq \emptyset$ for all $\delta \in (0,1]$.

Membership functions obeying the conditions (1) and (2) are called *characterizing functions*. If all δ-cuts of a fuzzy number are compact intervals, the corresponding fuzzy number is called *fuzzy interval*.

Remark 2.4 Methods for obtaining the characterizing function of fuzzy measurement data can be found in (Klir 1995, Viertl 2011 and Kovarova 2015).

For multivariate data and their statistical inference, the following concept of fuzzy vectors is necessary:

Definition 2.3 A fuzzy subset $\mathbf{x}^*$ of the Euclidean space $\mathbb{R}^n$ is called n-dimensional *fuzzy vector* if the membership function $\zeta(\cdot)$ of $\mathbf{x}^*$ fulfils the following:

(1) supp$[\zeta(\cdot)]$ is a bounded set, *i.e.* it is contained in an n-dimensional interval $\times_{i=1}^{n} [a_i, b_i]$ of finite volume.

(2) $C_\delta[\zeta(\cdot)]$ is non-empty for all $\delta \in (0,1]$, and it is a finite union of simply connected and closed subsets of $\mathbb{R}^n$.

Remark 2.5 A vector $(x_1^*, \dots, x_n^*)$ of fuzzy numbers is not a fuzzy vector. In this case, it is necessary to combine the fuzzy numbers $x_1^*, \dots, x_n^*$ to obtain a fuzzy element $(x_1, \dots, x_n)^*$ of the sample space $M_x^n \in \mathbb{R}^n$ and, then, the following holds:

Lemma 2.3 Let $x_1^*, \dots, x_n^*$ be fuzzy numbers with corresponding characterizing functions $\xi_1(\cdot), \dots, \xi_n(\cdot)$. Then the function $\zeta(\cdot, \dots, \cdot)$, defined by $\zeta(x_1, \dots, x_n) = min\{\xi_1(x_1), \dots, \xi_n(x_n)\}$ $\forall (x_1, \dots, x_n) \in \mathbb{R}^n$, is the vector-characterizing function of an n-dimensional fuzzy vector $(x_1, \dots, x_n)^*$.

Proof: By the validity of $C_\delta[\zeta(\cdot, \dots, \cdot)] = \times_{i=1}^{n} C_\delta[\xi_i(\cdot)]$, the δ-cut $C_\delta[\zeta(\cdot, \dots, \cdot)] \neq \emptyset \quad \forall \delta \in (0,1]$. Moreover, this δ-cut is a finite union of Cartesian products of compact intervals and the supp$[\zeta(\cdot, \dots, \cdot)]$ is contained in $\times_{i=1}^{n}$ supp$[\xi_i(\cdot)]$, as a result, the proof is concluded. Finally, if all δ-cuts of an n-dimensional fuzzy vector are simply connected, then the fuzzy vector is called n-dimensional fuzzy interval.

2.2. Fuzzy Probability Densities

Prior information in Bayesian inference is usually assumed in form of probability distributions on the parameter space Θ of stochastic models $X \sim f(\cdot|\theta)$; $\theta \in \Theta$. In case of fuzzy data, it turns out that a more general concept of probability is necessary. This leads to so-called fuzzy probability densities.

Definition 2.4 Let $(M, \mathcal{B}, \mu)$ be a measure space and $f^*: M \to \mathcal{F}_I(\mathbb{R}_+)$ be a function which assigns to $x \in M$ a fuzzy interval $f^*(x)$ for all $x \in M$, such that the so-called δ-level functions $\underline{f}_\delta(\cdot)$ and $\overline{f}_\delta(\cdot)$, defined by $C_\delta[f^*(x)] = [\underline{f}_\delta(x), \overline{f}_\delta(x)]$ $\forall x \in M$, $\forall \delta \in (0,1]$, are integrable functions, and there exists some classical probability density $g: M \to \mathbb{R}_+$ obeying $\underline{f}_1(x) \le g(x) \le \overline{f}_1(x)$ $\forall x \in M$, then f^* is called *fuzzy probability density*.

Based on fuzzy probability densities, a generalized concept of probability for event B in $\mathcal{B}$ can be defined, *i.e.* these generalized probabilities are fuzzy intervals $P^*(B)$, whose generating families are defined in the following way:

Definition 2.5 Based on a fuzzy probability density f^*, for $\delta \in (0,1]$, the system of classical probability densities g obeying $\underline{f}_\delta(x) \le g(x) \le \overline{f}_\delta(x)$ $\forall x \in M$ is denoted by $\mathcal{D}_\delta$. Then for an event B, the fuzzy probability $P^*(B)$ has generating family of intervals, $([a_\delta, b_\delta];\ \delta \in (0,1])$, given by

$$a_\delta := \inf\{\int_B g\ d\mu : g \in \mathcal{D}_\delta\} \text{ and } b_\delta := \sup\{\int_B g\ d\mu : g \in \mathcal{D}_\delta\}.$$

The characterizing function of the fuzzy interval $P^*(B)$ is given by the construction Lemma 2.2.

Remark 2.6 Fuzzy probability densities are basic for Bayesian inference with fuzzy data (Viertl and Sunanta 2013).

2.3. Integration of Fuzzy-valued Functions

Let $(M, \mathcal{B}, \mu)$ be a measure space and f^* a fuzzy valued function, where all values $f^*(x)$ are fuzzy intervals, and all δ-level functions be μ-integrable with finite integral. Then, it is possible to integrate f^* and this integral is a fuzzy interval, defined in the following way:

Definition 2.6 The fuzzy valued integral $J^* = \int_M f^* d\mu$ is the fuzzy interval, which has generating family $(A_\delta;\ \delta \in (0,1])$ where $A_\delta = [\underline{J}_\delta, \overline{J}_\delta]$, given by $\underline{J}_\delta := \int_M \underline{f}_\delta d\mu$ and $\overline{J}_\delta := \int_M \overline{f}_\delta d\mu$. The characterizing function $\eta(\cdot)$ of J^* is given by $\eta(x) = \sup\{\delta . \mathbb{1}_{[\underline{J}_\delta, \overline{J}_\delta]}(x): \delta \in [0,1]\}$ $\forall x \in \mathbb{R}$.

Remark 2.7 The integration of fuzzy valued functions is basic for the generalization of predictive distributions in Bayesian inference.

3. Fuzzy Samples and Foundations of Statistical Inference

Samples of continuous stochastic quantities X consist of a finite sequence of fuzzy numbers. In order to generalize statistical methods for fuzzy data, functions of data are important. Let M_X be the observation space of X, i.e. the set of possible values for X. Then the *sample space* for standard samples is the Cartesian product of n copies of the observation space M_X, *i.e.* the sample space is M_X^n. Classical statistical functions are mappings from the sample space to some measurable space $(N, \mathcal{A})$. For a classical statistical function $f: M \to N$, in case of fuzzy sample $x_1^*, \dots, x_n^*$, the generalized value $f(x_1^*, \dots, x_n^*)$ has to be defined in a reasonable way. This is possible by application of the so-called *extension principle* (see also Zadeh 1975) from the theory of fuzzy sets.

Definition 3.1 (*Extension principle*) Let $f: M \to N$ be any function, and x^* a fuzzy subset of M with membership function $\xi(\cdot)$. Then, the generalized value $f(x^*)$ is the fuzzy subset of N whose membership function $\eta(\cdot)$ is defined in the following way:

$$\eta(y) = \begin{cases} \sup\{\xi(x): x \in M, f(x) = y\} & \text{if } \exists x \in M: f(x) = y \\ 0 & \text{if } \nexists x \in M: f(x) = y \end{cases} \quad \forall y \in N$$

Remark 3.1 For $N = \mathbb{R}$, $\eta(\cdot)$ need not be a characterizing function. However, for continuous functions f, the following theorem holds:

Theorem 3.1 Let $f: \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $\mathbf{x}^*$ be an n-dimensional fuzzy vector. Then, the generalized value $f(x^*)$, obtained by application of the extension principle, is a fuzzy number. A detailed proof can be found in (Viertl 2011).

Remark 3.2 In order to apply the extension principle to functions of fuzzy samples, first, the vector of fuzzy data has to be combined into a fuzzy vector. This is done by the application of Lemma 2.3, *i.e.* the fuzzy sample with corresponding characterizing function $\xi_1(\cdot), \dots, \xi_n(\cdot)$ is combined into a fuzzy vector $\mathbf{x}^*$ whose vector-characterizing function $\zeta(\cdot, \dots, \cdot)$ has values

$$\zeta(x_1, \dots, x_n) = min\{\xi_1(x_1), \dots, \xi_n(x_n)\} \quad \forall (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Based on this combined fuzzy sample $\mathbf{x}^*$, statistical functions $\mathcal{S}: M_X^n \to N$ can be generalized by application of the extension principle. The fuzzy value $\mathcal{S}(x_1^*, \dots, x_n^*)$ is defined by $\mathcal{S}(\mathbf{x}^*)$, *i.e.* the membership function $\eta(\cdot)$ of $\mathcal{S}(x_1^*, \dots, x_n^*)$ is given by its values

$$\eta(y) = \begin{cases} sup\{\zeta(\mathbf{x}): \mathbf{x} \in M_X^n, \mathcal{S}(\mathbf{x}) = y\} & \text{if } \exists \mathbf{x} \in M: \mathcal{S}(\mathbf{x}) = y \\ 0 & \text{if } \nexists \mathbf{x} \in M: \mathcal{S}(\mathbf{x}) = y \end{cases} \quad \forall y \in N.$$

Example 3.1 For the sample mean $\overline{x} = \mathcal{S}(x_1, \dots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$ in case of n fuzzy numbers $x_1^*, \dots, x_n^*$, the fuzzy value $\overline{x}^* = \mathcal{S}(x_1^*, \dots, x_n^*)$ is also a fuzzy number by Theorem 3.1. Moreover, the generalized sample dispersion s^* is a fuzzy number. In figure 3.1, the characterizing functions of a fuzzy sample are depicted. In figure 3.2, the characterizing function of the corresponding fuzzy sample mean is given, along with the characterizing function of the fuzzy sample dispersion in figure 3.3 respectively.

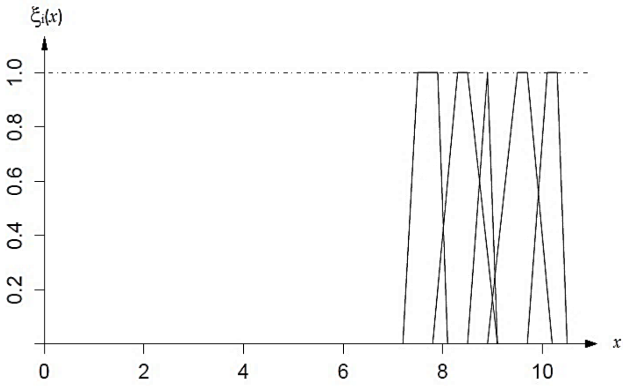

Figure 3.1. Fuzzy sample

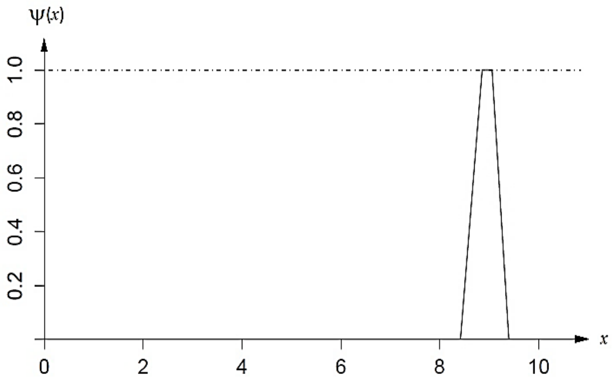

Figure 3.2. Fuzzy sample mean

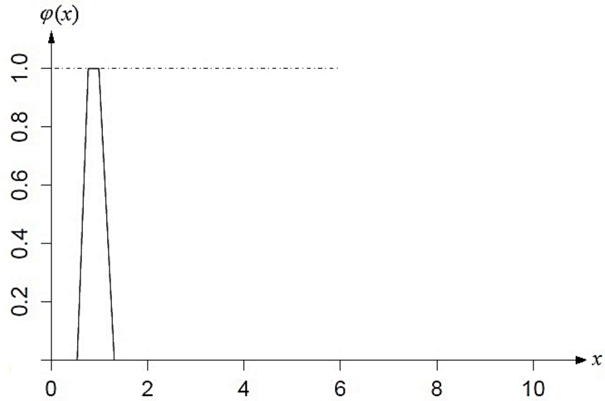

Figure 3.3. Fuzzy sample dispersion

4. Descriptive Statistics for Fuzzy Data

For fuzzy data, descriptive statistics has to be generalized. Mean values and empirical variances of fuzzy data are already introduced in Example 3.1.

4.1 Maxima and minima of fuzzy data

For fuzzy data $x_1^*, \dots, x_n^*$ with characterizing functions $\xi_1(\cdot), \dots, \xi_n(\cdot)$ and δ-cuts $C_\delta[\xi(\cdot)] = \bigcup_{j=1}^{k_{i,\delta}} [a_{i,\delta,j}, b_{i,\delta,j}]$, the fuzzy numbers x_{max}^* and x_{min}^* are defined by application of the extension principle for the functions $max\{x_1, \dots, x_n\}$ and $min\{x_1, \dots, x_n\}$ to the fuzzy combined sample **x*** (see also remark 3.2).

Definition 4.1 Let $x_1^*, \dots, x_n^*$ be *n* fuzzy numbers of the observation space $M_x \subseteq \mathbb{R}$ with corresponding characterizing functions $\xi_1(\cdot), \dots, \xi_n(\cdot)$. To obtain a fuzzy vector **x***, the fuzzy numbers $x_1^*, \dots, x_n^*$ have to be combined. Through construction of an *n*-dimensional vector-characterizing function $\zeta(\cdot, \dots, \cdot)$ via a triangular norm (t-norm *T*), the combined fuzzy sample **x*** forms a fuzzy element $(x_1, \dots, x_n)^*$ of the sample space M_x^n, *i.e.* $x_i^* \triangleq \xi_i(\cdot), i = 1(1)n \xRightarrow{t-norm\ T}$ combined fuzzy sample **x*** and vector-characterizing function $\zeta(\cdot, \dots, \cdot)$ where $\zeta(x_1, \dots, x_n) := T_n[\xi_1(x_1), \dots, \xi_n(x_n)] \quad \forall (x_1, \dots, x_n) \in \mathbb{R}^n$ and the combination T_n, which is the *n*-dimensional extension of the t-norm *T* by its associativity, *i.e.* $T_n(y_1, \dots, y_n) = T(y_1, T(\dots, T(y_{n-1}, y_n) \dots)) \qquad \forall (y_1, \dots, y_n) \in [0,1]^n$.

For statistical and algebraic calculations with fuzzy data, the minimum t-norm *T* is optimal (Viertl 2011), *i.e.* $\zeta(x_1, \dots, x_n) = T_n(\xi_1(x_1), \dots, \xi_n(x_n)) = min\{\xi_1(x_1), \dots, \xi_n(x_n)\} \qquad \forall (x_1, \dots, x_n) \in \mathbb{R}^n$.

Lemma 4.1 A fuzzy vector **x*** is obtained via minimum t-norm when the individual values of the variables x_i are fuzzy numbers x_i^*. Through the minimum-t-norm, the combination of *n* fuzzy numbers with characterizing functions $\xi_i(\cdot)$, $i = 1(1)n$, a fuzzy vector **x*** = $(x_1, \dots, x_n)^*$ is obtained. In this case, the following holds:

$$C_\delta[\zeta(\cdot, \dots, \cdot)] = \mathsf{X}_{i=1}^n C_\delta[\xi_i(\cdot)] \qquad \forall \delta \in (0,1]$$

In words, the δ-cuts of the fuzzy vector **x*** = $(x_1, \dots, x_n)^*$ are the Cartesian products of the δ-cuts of the fuzzy numbers x_i^*, *i* =1(1) *n*.

Proof:

$$\begin{aligned} C_\delta[\zeta(\cdot, \dots, \cdot)] &= \{\mathbf{x} \in \mathbb{R}^n : \zeta(\mathbf{x}) \geq \delta\} \\ &= \{\mathbf{x} : min\{\xi_1(x_1), \dots, \xi_n(x_n)\} \geq \delta\} \\ &= \{\mathbf{x} = (x_1, \dots, x_n) : \xi_i(x_i) \geq \delta \quad \forall i = 1(1)n\} \\ &= \mathsf{X}_{i=1}^n C_\delta[\xi_i(\cdot)] \end{aligned}$$

The concepts of combined fuzzy samples and triangular norms are useful for succinct multivariate statistical analysis of fuzzy data.

4.2 Histograms for fuzzy data

For a given partition of the observation space and fuzzy data in some cases, it is not possible to decide to which class a fuzzy observation belongs. This makes it necessary to generalize the concept of histograms. A first step is to construct lower and upper values of the frequencies $h_n(K_j)$ of class K_j, $j = 1(1)k$.

Let $x_1^*, \dots, x_n^*$ be fuzzy data with corresponding characterizing functions $\xi_1(\cdot),\dots,\xi_n(\cdot)$. Then, based on $\mathrm{supp}[\xi_i(\cdot)]$, the lower value $\underline{h}_n(K_j)$ is determined by

$$\underline{h}_n(K_j) := \frac{\{\#x_i^* : \mathrm{supp}[\xi_i(\cdot)] \subseteq K_j\}}{n} \quad \forall j = 1(1)k,$$

And the upper value $\overline{h}_n(K_j)$ is defined by

$$\overline{h}_n(K_j) := \frac{\{\#x_i^* : \mathrm{supp}[\xi_i(\cdot)] \cap K_j \neq \emptyset\}}{n},$$

where # indicates cardinality. Consequently, an interval valued histogram is obtained, whose frequencies are the intervals $h_n(K_j) = [\underline{h}_n(K_j), \overline{h}_n(K_j)]$ for $j = 1(1)k$.

Remark 4.1 For two disjoint classes K_j and K_l, the lower values of the frequencies are super-additive, *i.e.* $\underline{h}_n(K_j \cup K_l) \geq \underline{h}_n(K_j) + \underline{h}_n(K_l)$, while the upper values are sub-additive, *i.e.* $\overline{h}_n(K_j \cup K_l) \leq \overline{h}_n(K_j) + \overline{h}_n(K_l)$.

This is, in fact, easily seen for $\underline{h}_n(\cdot)$ by

$$\#x_i^* : \mathrm{supp}[\xi_i(\cdot)] \subseteq [K_j \cup K_l] \geq \#x_i^* : \mathrm{supp}[\xi_i(\cdot)] \subseteq K_j + \#x_i^* : \mathrm{supp}[\xi_i(\cdot)] \subseteq K_l,$$

and for $\overline{h}_n(\cdot)$ by

$$\#x_i^* : \mathrm{supp}[\xi_i(\cdot)] \cap [K_j \cup K_l] \neq \emptyset \leq \#x_i^* : \mathrm{supp}[\xi_i(\cdot)] \cap K_j \neq \emptyset + \#x_i^* : \mathrm{supp}[\xi_i(\cdot)] \cap K_l \neq \emptyset.$$

In Figure 4.1, an example of an interval-valued histogram is depicted.

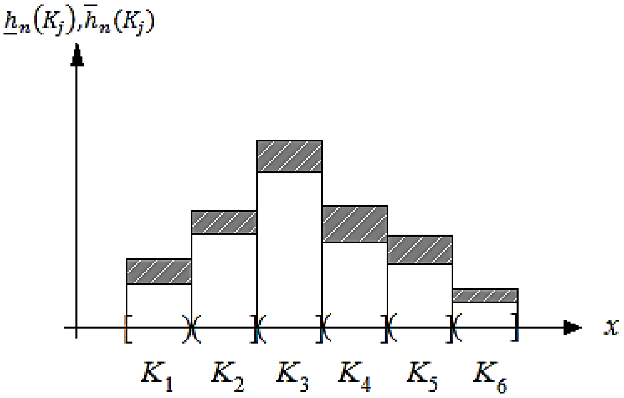

Figure 4.1. Interval-valued Histogram

Remark 4.2 Similar to the conditions for classical relative frequencies, whose abstraction leads to the axioms of probability distributions, the conditions of interval-valued frequencies are abstracted as axioms for so-called *interval probabilities*.

A more informative generalization of histograms is obtained when the considerations above are made for each δ-level. Then the results are fuzzy numbers $h_n^*(K_j)$ as fuzzy frequencies. The construction is the following:

For each $\delta \in (0,1]$, the generating family of intervals for the characterizing function of the fuzzy frequency $h_n^*(K_j)$ is $A_{j,\delta} = [\underline{h}_{n,\delta}(K_j), \overline{h}_{n,\delta}(K_j)]$, where $\underline{h}_{n,\delta}(K_j) := \frac{\{\#x_i^* : C_\delta[\xi_i(\cdot)] \subseteq K_j\}}{n}$ and $\overline{h}_{n,\delta}(K_j) := \frac{\{\#x_i^* : C_\delta[\xi_i(\cdot)] \cap K_j \neq \emptyset\}}{n}$.

By application of the construction lemma 2.2, the characterizing function of $h_n^*(K_j)$ is obtained.

Remark 4.3 The characterizing functions of fuzzy relative frequencies are step functions. Moreover, fuzzy histograms are examples of fuzzy valued functions:

$$f^*(x) := h_n^*(K_j) \ \forall x \in K_j$$

Similar to the interval-valued relative frequencies, i.e. for the generating values $\underline{h}_{n,\delta}(K_j)$ and $\overline{h}_{n,\delta}(K_j)$, the following holds: For $K_j \cap K_l = \emptyset$,

$$\underline{h}_{n,\delta}(K_j \cup K_l) \geq \underline{h}_{n,\delta}(K_j) + \underline{h}_{n,\delta}(K_l)$$
$$\overline{h}_{n,\delta}(K_j \cup K_l) \leq \overline{h}_{n,\delta}(K_j) + \overline{h}_{n,\delta}(K_l).$$

Figure 4.2 displayed the axonometric picture of a fuzzy histogram (Viertl 2011).

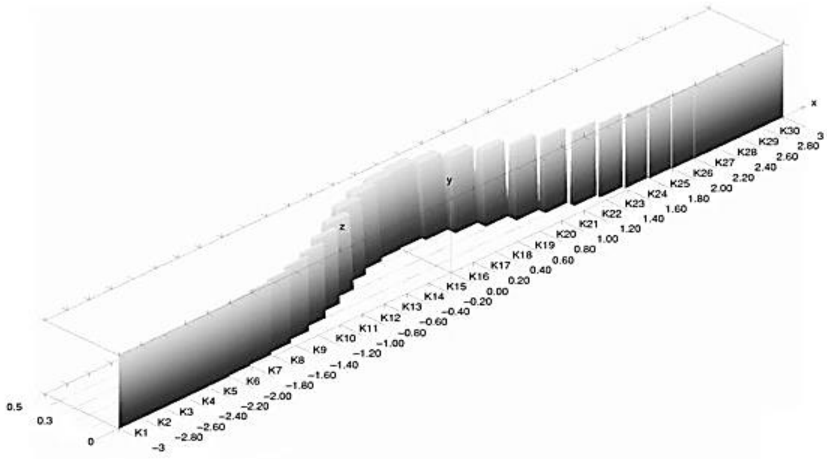

Figure 4.2. Fuzzy Histogram

4.3 Empirical distribution functions for fuzzy data

In order to generalize the empirical distribution function $\hat{F}_n(\cdot)$ to the situation of fuzzy data $x_1^*, \dots, x_n^*$ with characterizing functions $\xi_1(\cdot), \dots, \xi_n(\cdot)$, the following construction, using δ-cuts is useful which yields a fuzzy valued function $\hat{F}_n^*(\cdot)$: For fixed $x \in \mathbb{R}$ and $\delta \in (0,1]$, we define

$$\hat{F}_{\delta,U}(x) := \frac{\#x_i^* : C_\delta[\xi_i(\cdot)] \cap (-\infty, x] \neq \emptyset}{n}$$

$$\hat{F}_{\delta,L}(x) := \frac{\#x_i^* : C_\delta[\xi_i(\cdot)] \subseteq (-\infty, x]}{n}.$$

The generating system of intervals for the fuzzy value $\hat{F}_n^*(x)$ is $A_\delta(x) = [\hat{F}_{\delta,L}(x), \hat{F}_{\delta,U}(x)],\ \delta \in (0,1]$. The characterizing function of $\hat{F}_n^*(x)$ is obtained through lemma 2.2.

For variable $x \in \mathbb{R}$, two functions $\hat{F}_{\delta,L}(\cdot)$ and $\hat{F}_{\delta,U}(\cdot)$ are obtained, which in turn determine the $\hat{F}_n^*(\cdot)$ for variable δ. Based on $\hat{F}_n^*(\cdot)$, empirical fractiles for the fuzzy empirical distribution can be defined.

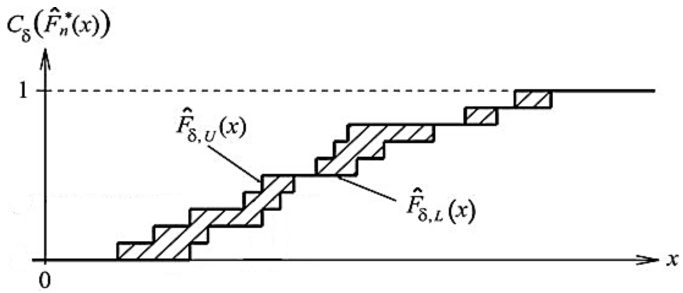

Figure 4.3. δ-Cut of $\hat{F}_n^*(\cdot)$

For $p \in (0,1)$, the lower and upper δ-level curves $\hat{F}_{\delta,L}(\cdot)$ and $\hat{F}_{\delta,U}(\cdot)$ are used to define the generating family of intervals $(A_\delta; \delta \in (0,1))$ for the fuzzy interval q_p^*, which is the empirical fractile of $\hat{F}_n^*(\cdot)$:

$$A_\delta := \left[\hat{F}_{\delta,U}^{-1}(p), \hat{F}_{\delta,L}^{-1}(p)\right] \quad \forall \delta \in (0,1)$$

where $\hat{F}_{\delta,U}^{-1}(p)$ and $\hat{F}_{\delta,L}^{-1}(p)$ are defined by

$\hat{F}_{\delta,U}^{-1}(p) := min\{x \in \mathbb{R}: \hat{F}_{\delta,U}(x) = p\}$ and $\hat{F}_{\delta,L}^{-1}(p) := max\{x \in \mathbb{R}: \hat{F}_{\delta,L}(x) = p\}$.

The characterizing function of the fuzzy fractile q_p^* is obtained from the construction lemma 2.2. An example for the definition of A_δ is given in Figure 3.4.

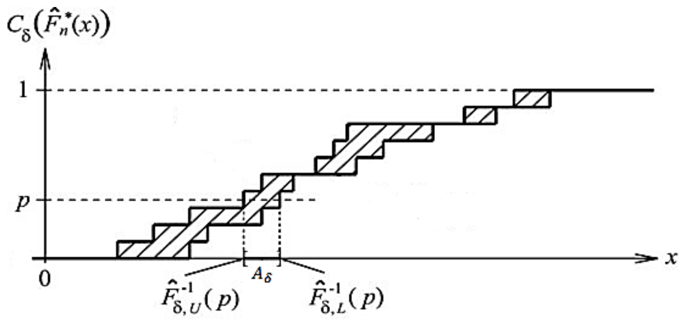

Figure 4.4. The definition of A_δ for q_p^*

4.4 Fuzzy empirical correlation

In case of fuzzy 2-dimensional data given as 2-dimensional fuzzy vector $(x_i, y_i)^*$, $i = 1(1)n$ with vector-characterizing functions $\zeta_i(\cdot,\cdot)$, the classical correlation coefficient can be generalized by application of the extension principle (Definition 3.1). First, the fuzzy vectors have to be combined into a fuzzy vector in the sample space $\mathbb{R}^{2n}$ by application of the minimum-t-norm, *i.e.* the vector-characterizing function $\zeta(\cdot,\ldots,\cdot)$ of the fuzzy combined sample is given by its values

$$\zeta(x_1, y_1, \ldots, x_n, y_n) := min\{\zeta_i(x_i, y_i): i = 1(1)n\} \quad \forall (x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2\cdot n}.$$

Now the extension principle is applied to the function

$$g(x_1, y_1, \ldots, x_n, y_n) = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}}$$

and the characterizing function $\psi_{r^*}(\cdot)$ of the generalized (fuzzy) empirical correlation coefficient r^* is given by its values

$$\psi_{r^*}(r) := \begin{Bmatrix} sup\{\zeta(x_1, y_1, \ldots, x_n, y_n): \text{for } g(x_1, y_1, \ldots, x_n, y_n) = r\} \\ 0 \text{ for } \nexists\ (x_1, y_1, \ldots, x_n, y_n): g(x_1, y_1, \ldots, x_n, y_n) = r \end{Bmatrix} \quad \forall r \in \mathbb{R}.$$

Remark 4.4 The support of $\psi_{r^*}(\cdot)$ is a subset of the interval [-1,1].

In figure 4.5, a fuzzy sample and the corresponding generalized empirical correlation coefficient r^* is given.

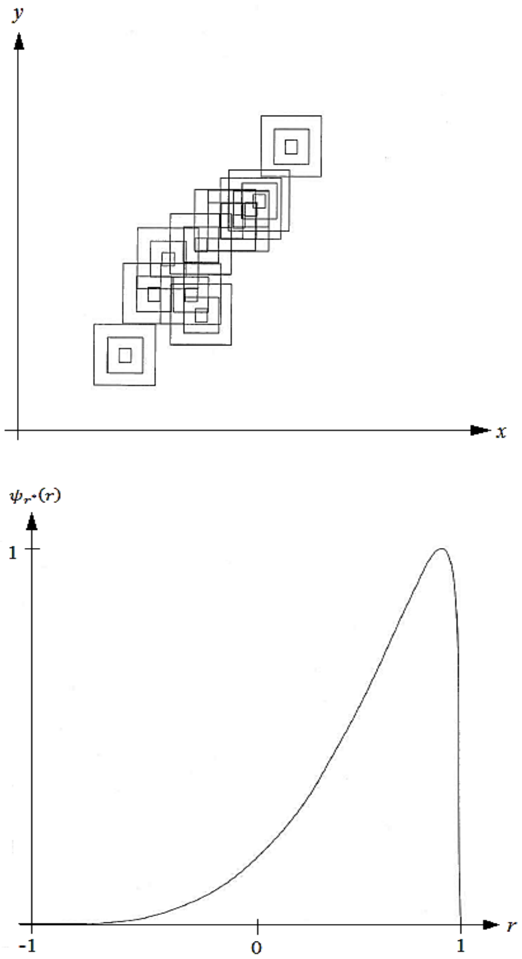

Figure 4.5. Fuzzy 2-dimensional data and its empirical correlation coefficient r^*

Remark 4.5 The definition of r^* allows different types of data. These data can be in form of fuzzy components x_i^* and y_i^*.

5. Objectivist Statistical Inference for Fuzzy Data

In objectivist statistics, it is assumed that a true underlying distribution exists which has to be estimated or tested. For parametric stochastic model $X \sim f(\cdot\,|\theta)$, $\theta \in \ominus$, the existence of a true value θ_0 of the parameter is assured, as a result, this θ_0 has to be estimated as good as possible.

5.1 Generalized Point Estimators

This is the generalization of so-called point estimators from standard statistics. For a stochastic quantity X with observation space M_X a certain characteristic value should be an element of N, where (N, ϱ) is a measurable space. Let $\tau: M_X^n \to N$ be a standard point estimator, which is a measurable function from the sample space M_X^n to the space of possible values of the characteristic value. This estimator is generalized by application of the extension principle in the following way:

Let $x_1^*, \dots, x_n^*$ with $x_i^* \mathrel{\widehat{=}} \xi_i(\cdot)$ be a fuzzy sample; first the vector-characterizing function $\zeta(\cdot, \dots, \cdot)$ of the combined fuzzy sample $\mathbf{x}^*$ is determined by its values

$$\zeta(x_1, \dots, x_n) := min\{\xi_1(x_1), \dots, \xi_n(x_n)\} \quad \forall (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Then, the membership function of the fuzzy estimate $\tau(x_1^*, \dots, x_n^*)$ is given by

$$\eta(s) = \begin{cases} sup\left\{min\{\xi_1(x_1), \dots, \xi_n(x_n)\}\right\} : \tau(x_1, \dots, x_n) = s \\ \quad 0 \quad \text{if } \nexists (x_1, \dots, x_n) \in \mathbb{R}^n : \tau(x_1, \dots, x_n) = s \end{cases} \quad \forall s \in N.$$

Remark 5.1 If the parameter θ_0 of a parametric stochastic model $X \sim f(\cdot\,|\theta)$ has to be estimated, a generalized estimator $\vartheta(x_1^*, \dots, x_n^*)$ is a fuzzy element of the parameter space $\ominus$.

Example 5.1 Let X be a stochastic quantity whose expectation $\mathbb{E}(X)$ exits. Then, for the fuzzy sample given in Figure 5.1, the fuzzy estimate for this sample is the generalization of the standard estimator $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. The characterizing function of the fuzzy estimate $\overline{x}^* = \frac{1}{n}\sum_{i=1}^{n} x_i^*$ is depicted in Figure 5.2.

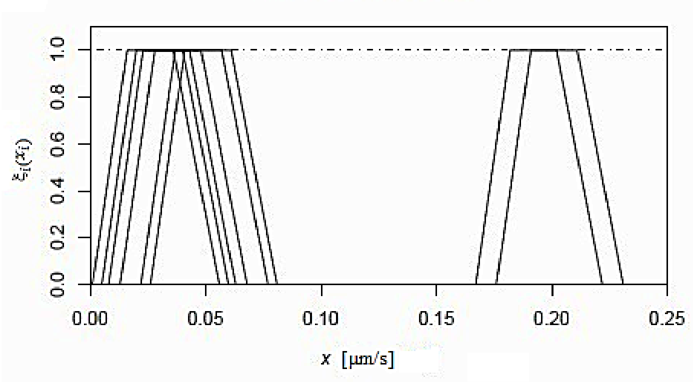

Figure 5.1. Characterizing functions of a fuzzy sample

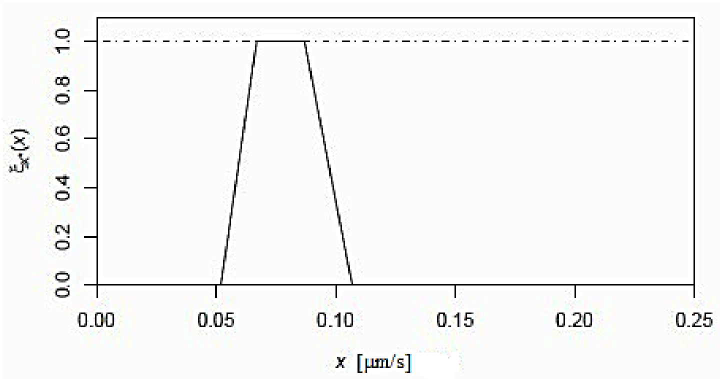

Figure 5.2. Characterizing function of the fuzzy sample mean

5.2 Generalized Confidence Sets

For a given parametric stochastic model $X \sim f_\theta$, $\theta \in \ominus$ and a standard confidence function $\boldsymbol{\kappa}: M_X^n \to \mathcal{P}(\ominus)$, where $\mathcal{P}(\ominus)$ denotes the power set of the parameter space $\ominus$, in case of fuzzy sample $x_1^*, \dots, x_n^*$, the concept of confidence set has to be generalized. This is possible in the following way:

Based on the combined fuzzy sample $\mathbf{x}^*$, with vector-characterizing function $\zeta(\cdot, \dots, \cdot)$, the membership function $\varphi(\cdot)$ of the generalized (fuzzy) confidence set $\ominus_{1-\alpha}^*$ is defined by

$$(\theta) = \begin{cases} sup\{\zeta(\mathbf{x}): \theta \in \boldsymbol{\kappa}(\mathbf{x})\} & \text{if } \exists \mathbf{x} \in M_X^n: \theta \in \boldsymbol{\kappa}(\mathbf{x}) \\ 0 & \text{if } \nexists \mathbf{x} \in M_X^n: \theta \in \boldsymbol{\kappa}(\mathbf{x}) \end{cases} \quad \forall \theta \in \ominus, \text{ where } \mathbf{x} = (x_1, \dots, x_n) \in M_X^n.$$

For the membership function $(\cdot)$, the following holds true:

$$I_{\bigcup_{\mathbf{x}:\zeta(\mathbf{x})=1} \boldsymbol{\kappa}(\mathbf{x})}(\theta) \le \varphi(\theta) \qquad \forall \theta \in \ominus,$$

which can be proven by observing $sup\{\zeta(\mathbf{x}): \theta \in \boldsymbol{\kappa}(\mathbf{x})\}$. Assuming $\theta \in \bigcup_{\mathbf{x}:\zeta(\mathbf{x})=1} \boldsymbol{\kappa}(\mathbf{x})$, then

$$\exists \mathbf{x} \in M_X^n: \theta \in \boldsymbol{\kappa}(\mathbf{x}) \Rightarrow \zeta(\mathbf{x}) = 1 \Rightarrow \varphi(\theta) = sup\{\zeta(\mathbf{x}): \theta \in \boldsymbol{\kappa}(\mathbf{x})\} = 1$$

$$\Rightarrow \varphi(\theta) \ge 1 \Rightarrow \varphi(\theta) \ge I_{\bigcup_{\mathbf{x}:\zeta(\mathbf{x})=1} \boldsymbol{\kappa}(\mathbf{x})}(\theta)$$

5.3 Statistical Tests based on Fuzzy Data

Classical test statistics $T = t(x_1, \dots, x_n)$ based on standard samples $x_1, \dots, x_n$ are measurable functions from the sample space M_X^n to a suitable measurable space (N, ϱ) which is partitioned into an *acceptance region* A and its complement $N \setminus A$, i.e. the *rejection region*.

For fuzzy samples $x_1^*, \dots, x_n^*$, the value $t^* = t(x_1^*, \dots, x_n^*)$ becomes fuzzy, and therefore it can be ambiguous to which region the value t^* belongs. Therefore, statistical tests have to be adapted accordingly.

A first solution would be to take observations until the support of t^* is a subset of A or A^c. Another method is to determine a *p*-value based on the fuzzy value t^*, where this *p*-value is defined as the probability of an error of the first type (rejecting a true hypothesis) for which the support of t^* is only contained in the rejection region A^c of the corresponding classical standard test.

Remark 5.2 For fuzzy samples, a more natural approach is the generalization of *p*-values in form of fuzzy numbers, as explained here:

Let $\eta(\cdot)$ be the characterizing function of the fuzzy value t^* of the generalized test statistic $t(x_1^*, \dots, x_n^*)$. Considering the δ-cuts of t^*, $C_\delta[t^*] = [t_1(\delta), t_2(\delta)]$ $\forall \delta \in (0,1]$, the fuzzy *p*-value p^* for a given standard test is defined in the following way:

For one-sided tests with test statistic T and decision rule "rejection for $T \le t_{critical}$", the generating family of intervals for the fuzzy *p*-value p^* is defined by

$$A_\delta := [Pr\{T \le t_1(\delta)\}, Pr\{T \le t_2(\delta)\}] \qquad \forall \delta \in (0,1].$$

For one-sided tests with decision rule "rejection for $T \ge t_{critical}$", the generating family $(A_\delta\,;\, \delta \in (0,1])$ for p^* is defined by

$$A_\delta := [Pr\{T \ge t_2(\delta)\}, Pr\{T \ge t_1(\delta)\}] \qquad \forall \delta \in (0,1].$$

In case of two-sided tests, *i.e.* acceptance for $t_l \le T \le t_u$, and fuzzy value t^* of the test statistic T, first, it has to be decided on which side, of the median *m* of the distribution of T, the main part of fuzziness of t^* is located. Therefore, the areas under the characterizing function $\eta(\cdot)$ of t^* have to be computed, which are on both sides of the median *m*. Denoting these areas by F_1 and F_2 respectively, the generating family $(A_\delta\,;\, \delta \in (0,1])$ for the fuzzy *p*-value p^* are defined by

$$A_\delta := \begin{Bmatrix} [2Pr\{T \le t_1(\delta)\}, min\{1, Pr\{T \le t_2(\delta)\}\}] \text{ if } F_1 > F_2 \\ [2Pr\{T \ge t_2(\delta)\}, min\{1, Pr\{T \ge t_1(\delta)\}\}] \text{ if } F_1 \le F_2 \end{Bmatrix} \forall \delta \in (0,1].$$

The δ-cuts of p^* are denoted by $[p_1(\delta), p_2(\delta)]$, and can be interpreted in terms of generalized probabilities, and can be compared with the significance level α of the test. The decision is made according to a three-decision testing problem:

If, for all $\delta \in (0,1]$ and $p_1(\delta) \le p_2(\delta)$,

$p_2(\delta) < \alpha$: reject $\mathcal{H}_0$ and accept $\mathcal{H}_1$

$p_1(\delta) > \alpha$: accept $\mathcal{H}_0$ and reject $\mathcal{H}_1$

$\alpha \in [p_1(\delta), p_2(\delta)]$: both $\mathcal{H}_0$ and $\mathcal{H}_1$ are neither accepted nor rejected.

In the third case, the uncertainty of making a decision is expressed by the characterizing function ξ(·) of p^*.

The case that $t_1(\delta) = t_2(\delta)$ for all $\delta \in (0,1]$ implies $p_1(\delta) = p_2(\delta)$, *i.e.* we have a two-decision problem similar to tests based on precise data (Filzmoser 2004).

Example 5.2 Let the test statistic T have a standard normal distribution with density $f(\cdot)$ and the characterizing function $\eta(\cdot)$ of $t^*(0.2, 0.7, 1.2)$ is of symmetric triangular shape as shown in Figure 4.3a. The following one-sided statistical hypotheses are to be tested at the significance level α = 0.05,

$\mathcal{H}_0$: $\theta \le \theta_0$ vs. $\mathcal{H}_1$: $\theta > \theta_0$; θ is an unknown parameter and $\theta_0 = 0$ in this example, *i.e.* $\mathcal{H}_0$: $\theta \le 0$ vs. $\mathcal{H}_1$: $\theta > 0$. The δ-cuts of the fuzzy *p*-value p^* as defined above are to be determined for decision.

In Figure 4.3a, the δ-cut of t^*, $C_\delta[t^*] = [t_1(\delta), t_2(\delta)]$, for δ = 0.5 is derived as $[t_1(0.5), t_2(0.5)] = [0.45, 0.95]$ accordingly. The *p*-values corresponding to $t_1(0.5)$ and $t_2(0.5)$ are

0.33 and 0.17 respectively as shown in shaded areas in Figure 4.3a and the construction of their characterizing function in Figure 4.3b. Finally, the resulting fuzzy *p*-value is compared to the significance level α = 0.05 and $\mathcal{H}_0$ conclude that is not rejected.

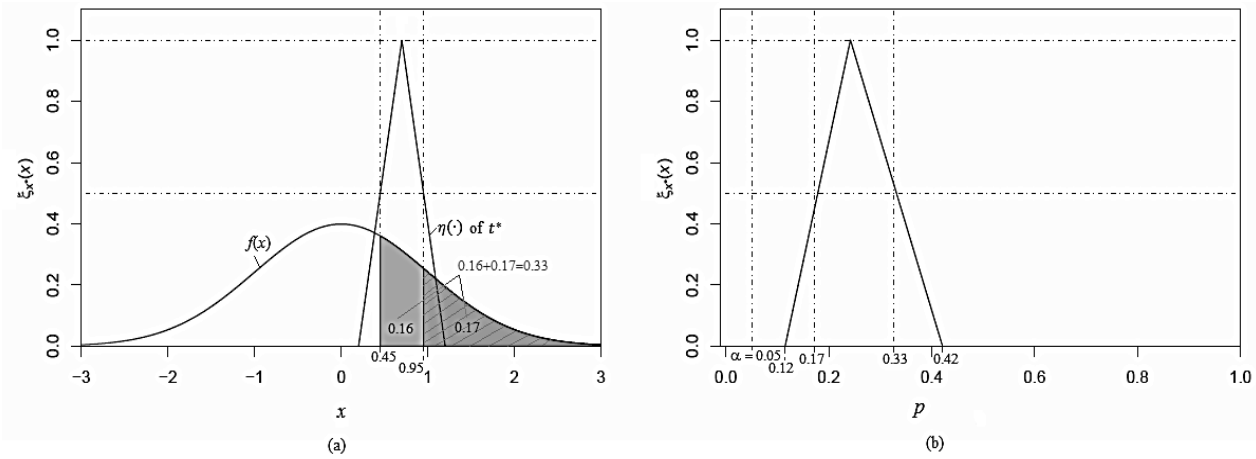

Figure 4.3: Construction of the Characterizing Function of the Fuzzy *p*-value

6. Bayesian Inference and Fuzzy Information

Bayesian inference uses a-priori information of specific parameters in stochastic models. In other words, Bayes' theorem formulates the transition from the a-priori distribution $\pi(\theta)$ of the stochastic quantity, describing parameters of interest, to the so-called a-posteriori distribution $\pi(\theta|D)$ based on data *D*. In case of continuous stochastic models $X \sim f(\cdot|\theta)$; $\theta \in \ominus$, based on observations $x_1, \dots, x_n$ of *X*, the transition from an a-priori density to an updated information with the distribution of the stochastic quantity describing the parameter θ is given by the conditional density $\pi(\cdot \mid x_1, \dots, x_n)$ of $\tilde{\theta}$, *i.e.* Bayes' theorem

$$\pi(\theta|x_1,\dots,x_n) = \frac{\pi(\theta)\cdot l(\theta\,;x_1,\dots,x_n)}{\int_{\ominus}\pi(\theta)\cdot l(\theta\,;x_1,\dots,x_n)d\theta} \text{ or } \pi(\theta|x_1,\dots,x_n)\ \alpha\ \pi(\theta)\cdot l(\theta\,;\,x_1,\dots,\,x_n) \quad \forall\theta\in\ominus,$$

where $l(\theta\,;\,x_1,\dots,\,x_n)$ is the likelihood function defined on the parameter space $\ominus$.

In standard Bayesian inference, it is assumed here that $\pi(\theta)$ is a standard (classical) probability density, and the data are given as numbers or vectors.

For a more realistic Bayesian statistical inference, the inevitable uncertainties (including fuzziness) have to be addressed, *i.e.* the fuzziness of continuous quantities and of the a-priori knowledge, however. As explained in Section 2, this fuzziness can be defined in form of fuzzy numbers or vectors and modelled by the so-called fuzzy probability densities, respectively. Finally, Bayes' theorem can be generalized to handle the situation of fuzzy a-priori density and fuzzy data.

6.1 Fuzzy a-priori densities

Based on fuzzy probability densities (Section 2.2), generalized fuzzy probabilities of events $A \in \mathcal{A}$, $P^*(A)$ are fuzzy intervals. The characterizing function $\xi(\cdot)$ of $P^*(A)$ is given by the generation lemma (Viertl 2011), *i.e.* $\xi(y) = sup\{\delta.I_{[a_\delta, b_\delta]}(x)\colon \delta\in(0,1]\}$ for all $y\in\mathbb{R}$, where $I_B(\cdot)$ denotes the indicator function of the set *B*, and $[a_0, b_0] := \mathbb{R}$.

Definition 6.1: A fuzzy a-priori density on the parameter space $\ominus$ is a fuzzy density defined on $\ominus$.

6.2 Generalized likelihood function

The likelihood function $l(\theta; x_1,\dots,x_n)$ in Bayes' theorem can be generalized for fuzzy data $x_1^*,\dots,x_n^*$ through the extension principle and based on the combined fuzzy sample $\mathbf{x}^*$ and its vector-characterizing function $\zeta(\cdot,\dots,\cdot)$. The characterizing function $\psi(\cdot)$ of the fuzzy value $l(\theta;\mathbf{x}^*)$ is, then, given by its values as

$$\psi_{l(\theta;\mathbf{x}^*)}(y) = \begin{Bmatrix} sup\{\zeta(\mathbf{x})\colon \mathbf{x}\in M_X^n \wedge l(\theta;\mathbf{x}) = y\} & \text{if } \exists\mathbf{x} : l(\theta;\mathbf{x}) = y \\ 0 & \text{if } \nexists\mathbf{x}\in M_X^n\colon l(\theta;\mathbf{x}) = y \end{Bmatrix} \quad \forall y\in\mathbb{R}.$$

Based on the generalized likelihood function, Bayes' theorem is generalized for fuzzy a-priori densities $\pi^*(\cdot)$ on the parameter space Θ and fuzzy data.

6.3 Generalized Bayes' theorem for fuzzy data

The following description of Bayes' theorem is generalized for the continuous case. However, for discrete variables, Bayes' formula can also be generalized similarly (Sunanta 2016). Let $X \sim f(\cdot|\theta)$; $\theta \in \Theta$ be a continuous stochastic model with continuous parameter space $\Theta \subseteq \mathbb{R}^k$, and $\pi^*(\cdot)$ a fuzzy a-priori density on the parameter space Θ. The sequential updating from standard Bayes' theorem is generalized as following:

Given a fuzzy a-priori density $\pi^*(\cdot)$ and fuzzy sample $x_1^*, \dots, x_n^*$, the generalized a-posteriori density $\pi^*(\cdot \mid x_1^*, \dots, x_n^*)$ is then calculated, from which the result is the same as if the sample were separated in two parts $x_1^*, \dots, x_m^*$ and $x_{m+1}^*, \dots, x_n^*$, *i.e.* the a-posteriori density of the first partition $\pi^*(\cdot \mid x_1^*, \dots, x_m^*)$ is obtained. Then, this a-posteriori density is taken as new a-priori density, based on which the a-posteriori density $\pi^*(\cdot \mid x_{m+1}^*, \dots, x_n^*)$ is finally calculated.

The generalized fuzzy Bayes' theorem is based on δ-level functions of the fuzzy a-priori density and of the generalized likelihood function, as well as the vector-characterizing function of the combined fuzzy sample $\mathbf{x}^*$.

Using the above notation, the δ-level functions of the fuzzy a-posteriori density $\pi^*(\cdot \mid x_1^*, \dots, x_n^*)$, with combined fuzzy sample $\mathbf{x}^*$, obtained by minimum-combination rule, are defined in the following way:

$$\overline{\pi}_\delta(\theta|\mathbf{x}^*) = \frac{\overline{\pi}_\delta(\theta)\cdot \overline{l}_\delta(\theta\,;\mathbf{x}^*)}{\int_\Theta \frac{1}{2}\left[\underline{\pi}_\delta(\theta)\cdot \underline{l}_\delta(\theta;\mathbf{x}^*) + \overline{\pi}_\delta(\theta)\cdot \overline{l}_\delta(\theta;\mathbf{x}^*)\right] d\theta}$$

and $\qquad \forall\, \theta \in \Theta$

$$\underline{\pi}_\delta(\theta|\mathbf{x}^*) = \frac{\underline{\pi}_\delta(\theta)\cdot \underline{l}_\delta(\theta\,;\mathbf{x}^*)}{\int_\Theta \frac{1}{2}\left[\underline{\pi}_\delta(\theta)\cdot \underline{l}_\delta(\theta\,;\mathbf{x}^*) + \overline{\pi}_\delta(\theta)\cdot \overline{l}_\delta(\theta\,;\mathbf{x}^*)\right] d\theta}$$

The averaging in the integral is necessary to keep the sequential updating condition (Viertl 2011).

6.4 Fuzzy predictive distribution

In Bayesian inference, the predictive density based on the a-posteriori density $\pi(\cdot|D)$ is defined to be the marginal density of X from $(X,\tilde{\theta})$ denoted by $p(\cdot|D)$, whose values are given by

$$p(x|D) := \int_\Theta f(x|\theta)\pi(\theta|D)\, d\theta \quad \forall x \in M_X.$$

The generalization for fuzzy a-posteriori densities $\pi^*(\cdot|\mathbf{x}^*)$ is obtained through a construction similar to the calculation of probabilities based on fuzzy densities (section 2.2).

Defining a continuous stochastic model $X \sim f(\cdot|\theta)$; $\theta \in \Theta$ and the fuzzy a-posteriori density $\pi^*(\cdot|\mathbf{x}^*)$, the generalized integral $\int_\Theta f(x|\theta)\pi^*(\theta|\mathbf{x}^*)\, d\theta \quad \forall x \in M_X$ is a fuzzy interval $p^*(x|\mathbf{x}^*)$ whose characterizing function is generated by the following family (A_δ; $\delta \in (0,1]$) of intervals A_δ:

For all $\delta \in (0,1]$, let $\underline{\pi}_\delta(\cdot|\mathbf{x}^*)$ and $\overline{\pi}_\delta(\cdot|\mathbf{x}^*)$ be the δ-level functions of the fuzzy a-posteriori density $\pi^*(\cdot|\mathbf{x}^*)$, and D_δ the set of all classical probability densities $g(\cdot)$ on Θ, where $\underline{\pi}_\delta(\theta|\mathbf{x}^*) \le g(\theta) \le \overline{\pi}_\delta(\theta|\mathbf{x}^*)$ $\forall\, \theta \in \Theta$.

The interval $A_\delta = [c_\delta, d_\delta]$, $\delta \in (0,1]$ is defined by

$c_\delta := inf\{\int_\Theta g(\theta)d\theta : g \in D_\delta\}$ and $d_\delta := sup\{\int_\Theta g(\theta)d\theta : g \in D_\delta\}$. The system of intervals $[c_\delta, d_\delta]$, $\delta \in (0,1]$, is nested by the validity of $\delta_1 < \delta_2 \Rightarrow D_{\delta_1} \supseteq D_{\delta_2}$.

The characterizing function of $p^*(x|\mathbf{x}^*)$ is given by the generation lemma from Section 6.1, *i.e.*

$\psi_x(y) = sup\{\delta . I_{[c_\delta, d_\delta]}(y) : \delta \in [0,1]\} \quad \forall y \in \mathbb{R}$, where $[c_0, d_0] := \mathbb{R}$.

The fuzzy predictive density is, then, the family of $p^*(x|\mathbf{x}^*)$, $x \in M_X$.

7 Fuzzy Stochastic Processes

Stochastic processes are families of stochastic quantities $X_t, t \in T$ where T is an appropriate index set, frequently T is a time interval, *i.e.* $T \subseteq \mathbb{R}$. The occurring of uncertainty can be modelled using the theory of *fuzzy random processes*, which is derived from the theory of fuzzy random functions (Möller 2009). By the imprecision of data from continuous quantities, the observations (trajectories) of the process are fuzzy. Therefore, the sample paths are fuzzy valued functions. For the so-called *fuzzy stochastic process* $X_t^*, t \in T$, different mathematical models exist, i.e. X_t^* can be modelled as *fuzzy random variables*.

A special case would be time series, in which case the uncertainty of the individual observed value is modeled as a fuzzy variable $X_n^*, n = 1(1)N$. A time series of fuzzy data may be viewed as a random realization of a *fuzzy stochastic process*, of which the generalized model is necessary for further forecasting (Möller and Reuter 2007). However, for fuzzy stochastic processes with continuous time, many challenging research topics are still open, *e.g.* the first hitting times, predictions, an analogue of convergence theorems.

Conclusions

For observations and measurements of continuous quantities, fuzziness is unavoidable. Therefore, suitable mathematical models are necessary to describe real data. This is possible and many research topics related to this are still waiting to be solved. The fuzziness of individual measurement results can be described by so-called fuzzy numbers, whereas the variability and errors are described by stochastic models.

In this contribution, some generalized statistical methods for fuzzy data, *i.e.* descriptive statistics, statistical inference, and Bayesian inference are described. Descriptive statistics provide simple summaries of the collected samples and measures (data). They form the basis of virtually every quantitative analysis of the data. Through concepts of fuzzy numbers and characterising functions, fuzzy data are summarised and represented in form of fuzzy histograms. Some other statistics, such as fuzzy empirical distribution functions and correlation coefficients, are also useful for preliminary data analysis. For realistic projection of the behaviours of the variables under analysis, models for prediction based on fuzzy information, through Bayesian inference and fuzzy predictive density, are introduced.

Fuzziness is everywhere in the physical world, including in economics arena. In order to describe different facets of reality, the analysis methods have to capture this type of uncertainty. The related methods are available through mathematical models for fuzzy data. Accordingly, application of such methods results in more realistic models for data analysis and, subsequently, better understanding of the collected data for further use of such information.

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